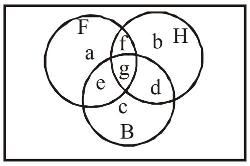
Hints and Solutions

Chapter-1 : Sets

1. (15) n(A – B) = n(A) – n(A Ç B) = 25 – 10 = 15.

2. (325)



$a + e + f + g = 285, \quad b + d + f + g = 195$

$c + d + e + g = 115, \quad e + g = 45, f + g = 70,$

$d + g = 50$

$a + b + c + d + e + f + g = 500 - 50 = 450$

We obtain

$a + f = 240, b + d = 125, \ c + e = 65$

$a + e = 215, b + f = 145; b + c + d = 165$

$a + c + e = 255; \ a + b + f = 335$

Solving we get

b = 95, c = 40, a = 190 , d = 30, e = 25, f = 50 and g = 20

Desired quantity = a + b + c = 325

3. (3) n(A È B) = n(A) + n(B) – n(A Ç B) = 12 + 9 – 4 = 17

Now, $n((A \cup B)^c) = n(U) - n(A \cup B) = 20 - 17 = 3.$

4. (41) Let R be the set of families having a radio and *T* the set families having a T.V.,

then $n(R \cup T)$ = The number of families having at least one of the radio and T.V. = 1003 – 63 = 940

n(R) = 794 and n(T) = 187

Let *x* families have both a radio and a T.V.

Then number of families who have only radio = 794 – *x*

And the number of families who have only T.V. = 187 – *x*

Now, $794 - x + x + 187 - x = 940$

$\Rightarrow 981 - x = 940$ or $x = 981 - 940 = 41$

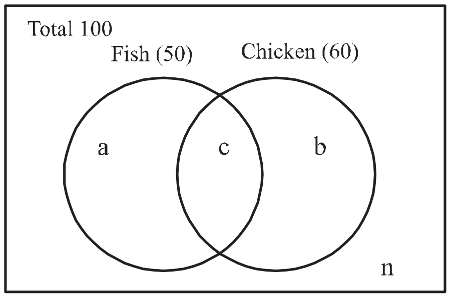
Hence, the required number of families having both a radio and a T.V. = 41

5. (100) ∵ Product of two even number is always even and product of two odd numbers is always odd.

∴ Number of required subsets = Total number of subsets – Total number of subsets having only odd numbers = $2^{100} - 2^{50}$

$= 2^{50}(2^{50} - 1)$

6. (20) Total number of persons = $a + b + c + n = 100$



Do not prefer fish $b + n = 50$

60 prefer chicken hence $b + c = 60$

Do not like fish and chicken is $n = 10$

On solving these equations we will get $a = 30, b = 40, c = 20$

The number of persons who prefer both fish and chicken $= c = 20$

7. (32768) Let $x \in A$, then

$\because 2^{(x+2)(x^2-5x+6)} = 1 \Rightarrow (x+2)(x-2)(x-3) = 0$

$x = -2, 2, 3$

$A = \{-2, 2, 3\}$

Then, $n(A) = 3$

Let $x \in B$, then

$-3 < 2x - 1 < 9$

$-1 < x < 5$ and $x \in Z$

$\therefore B = \{0, 1, 2, 3, 4\}$

$n(B) = 5$

$n(A \times B) = 3 \times 5 = 15$

Hence, number of subsets of $A \times B = 2^{15}$

8. (300) *n* (A' Ç B') = *n* (A È B)' = *n* (U) – *n* (A È B)

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 700 - [200 + 300 - 100] = 300$

Chapter-2 : Relations and Functions - I

1. (9) A × B = {(2, 7), (2, 8), (2, 9), (4, 7), (4, 8), (4, 9), (5, 7), (5, 8), (5, 9)}

$n(A \times B) = n(A) \,.\, n(B) = 3 \times 3 = 9$

2. (4) We have, $f(x) = \left[\log_{10}\left(\frac{5x - x^2}{4}\right)\right]^{1/2}$ .....(i)

From (i), clearly f(x) is defined for those values of x for which $\log_{10}\left[\frac{5x - x^2}{4}\right] \geq 0$

$\Rightarrow \left(\frac{5x - x^2}{4}\right) \geq 10^0 \Rightarrow \left(\frac{5x - x^2}{4}\right) \geq 1$

$\Rightarrow x^2 - 5x + 4 \leq 0 \Rightarrow (x-1)(x-4) \leq 0$

Hence, domain of the function is [1, 4] i.e. $1 \le x \le 4$.

3. (0.5) $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10,10) \Rightarrow f(x) = \log\left(\frac{10+x}{10-x}\right)$

$$\Rightarrow f\left(\frac{200x}{100+x^2}\right) = \log\left[\frac{10+\frac{200x}{100+x^2}}{10-\frac{200x}{100+x^2}}\right]$$

$$= \log\left[\frac{10(10+x)^2}{10(10-x)^2}\right] = 2\log\left(\frac{10+x}{10-x}\right) = 2f(x)$$

$$\therefore f(x) = \frac{1}{2}f\left(\frac{200x}{100+x^2}\right) \Rightarrow k = \frac{1}{2} = 0.5.$$

4. (6) $A - B = \{1, 2, 3\} - \{4, 5, 6\} = \{1, 2, 3\}$

$A \cap C = \{1, 2, 3\} \cap \{1, 2\} = \{1, 2\}$

$\therefore (A - B) \times (A \cap C) = \{1, 2, 3\} \times \{1, 2\}$

$= \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$

5. (3) When x = 1, y = 7 ∈ N, so (1, 7) ∈ R

When x = 2, y = 2 + 3 = 5 ∈ N, so (2, 5) ∈ R

Again for x = 3, y = 3 + 2 = 5 ∈ N, (3, 5) ∈ R

Similarly for x = 4, $y = 4 + \frac{6}{4} \notin N$ and for x = 5, $y = 5 + \frac{6}{5} \notin N$. Thus, R = {(1, 7), (2, 5), (3, 5)}

∴ Domain of R = {1, 2, 3} and Range of R = {7, 5}.

6. (3) R be a relation on N defined by $x + 2y = 8$

∴ R = {(2, 3); (4, 2); (6, 1)}

Hence, domain of R = {2, 4, 6}

7. (4) $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$

For x = 7, $3f(7) + 2f(11) = 70 + 30 = 100$

For x = 11, $3f(11) + 2f(7) = 140$

$$\frac{f(7)}{-20} = \frac{f(11)}{-220} = \frac{-1}{9-4} \Rightarrow f(7) = 4.$$

8. (512) A = {2, 4, 6 }, B = { 2, 3, 5}

Number of relations from A to B = $2^{3 \times 3} = 2^9$

Chapter-3 : Trigonometric Functions

1. (3.25) $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$

$= 3(1 - 2\sin\theta\cos\theta)^2 + 6(1 + 2\sin\theta\cos\theta) + 4\sin^6\theta$

$= 3(1 + 4\sin^2\theta\cos^2\theta - 4\sin\theta\cos\theta) + 6 + 12\sin\theta\cos\theta + 4\sin^6\theta$

$= 9 + 12\sin^2\theta\cos^2\theta + 4\sin^6\theta$

$= 9 + 12\cos^2\theta(1 - \cos^2\theta) + 4(1 - \cos^2\theta)^3$

$= 9 + 12\cos^2\theta - 12\cos^4\theta + 4(1 - \cos^6\theta - 3\cos^2\theta + 3\cos^4\theta)$

$= 9 + 4 - 4\cos^6\theta$

$= 13 - 4\cos^6\theta$

2. (2) $\sin x - \sin 2x + \sin 3x = 0$

$\Rightarrow \sin x - 2\sin x.\cos x + 3\sin x - 4\sin^3 x = 0$

$\Rightarrow 4\sin x - 4\sin^3 x - 2\sin x.\cos x = 0$

$\Rightarrow 2\sin x(1 - \sin^2 x) - \sin x.\cos x = 0$

$\Rightarrow 2\sin x.\cos^2 x - \sin x.\cos x = 0$

$\Rightarrow \sin x.\cos x(2\cos x - 1) = 0$

$\therefore \sin x = 0, \cos x = 0, \cos x = \frac{1}{2}$

$\therefore x = 0, \frac{\pi}{3} \quad \because x \in \left[0, \frac{\pi}{2}\right)$

3. (9) $A = \cos\frac{\pi}{2^2}.\cos\frac{\pi}{2^3} \ldots \cos\frac{\pi}{2^{10}}.\sin\frac{\pi}{2^{10}}$

$$= \frac{1}{2}\left(\cos\frac{\pi}{2^2}.\cos\frac{\pi}{2^3} \ldots \cos\frac{\pi}{2^9}\sin\frac{\pi}{2^9}\right)$$

$$= \frac{1}{2^8}\left(\cos\frac{\pi}{2^2}.\sin\frac{\pi}{2^2}\right) = \frac{1}{2^9}\sin\frac{\pi}{2}$$

$$= \frac{1}{512} = \frac{1}{2^9}$$

4. (19) Let, the functions is,

$$f(\theta) = 3\cos\theta + 5\sin\theta\cdot\cos\frac{\pi}{6} - 5\sin\frac{\pi}{6}\cos\theta$$

$$= 3\cos\theta + 5\times\frac{\sqrt{3}}{2}\sin\theta - 5\times\frac{1}{2}\cos\theta$$

$$= \left(3 - \frac{5}{2}\right)\cos\theta + 5\times\frac{\sqrt{3}}{2}\sin\theta$$

$$= \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$$

$$\max f(\theta) = \sqrt{\frac{1}{4} + \frac{25}{4}\times 3} = \sqrt{\frac{76}{4}} = \sqrt{19}$$

5. (2) ∵ The given equation is

$\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cdot\cos\beta$, $\alpha, \beta \in [0, \pi]$

Then, by A.M., G.M. ineqality;

A.M. ≥ G.M.

$$\frac{\sin^4\alpha + 4\cos^4\beta + 1 + 1}{4} \ge \left(\sin^4\alpha\cdot 4\cos^4\beta\cdot 1\cdot 1\right)^{\frac{1}{4}}$$

$\sin^4\alpha + 4\cos^4\beta + 1 + 1 \ge 4\sqrt{2}\sin\alpha\cdot|\cos\beta|$

Inequality still holds when $\cos\beta < 0$ but L.H.S. is positive than $\cos\beta > 0$, then

L.H.S. = R.H.S

$\therefore \sin^4\alpha = 1$ and $\cos^4\beta = \frac{1}{4}$

$\Rightarrow \alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{4}$

$\therefore \cos(\alpha + \beta) - \cos(\alpha - \beta)$

$$= \cos\left(\frac{\pi}{2} + \beta\right) - \cos\left(\frac{\pi}{2} - \beta\right)$$

$$= -\sin\beta - \sin\beta = -2\sin\frac{\pi}{4} = -\sqrt{2}$$

6. (0.75) $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$

$$= \left(\frac{1+\cos 20°}{2}\right) + \left(\frac{1+\cos 100°}{2}\right) - \frac{1}{2}(2\cos 10° \cos 50°)$$

$$= 1 + \frac{1}{2}(\cos 20° + \cos 100°) - \frac{1}{2}[\cos 60° + \cos 40°]$$

$$= \left(1 - \frac{1}{4}\right) + \frac{1}{2}[\cos 20° + \cos 100° - \cos 40°]$$

$$= \frac{3}{4} + \frac{1}{2}[2\cos 60° \times \cos 40° - \cos 40°]$$

$$= \frac{3}{4}$$

7. (5) Consider equation, $1 + \sin^4 x = \cos^2 3x$

L.H.S. = $1 + \sin^4 x$ and R.H.S. = $\cos^2 3x$

∵ L.H.S. ≥ 1 and R.H.S. ≤ 1 ⇒ L.H.S. = R.H.S. = 1

$\sin^4 x = 0$, and $\cos^2 3x = 1$

⇒ $\sin x = 0$ and $(4\cos^2 x - 3)^2 \cos^2 x = 1$

⇒ $\sin x = 0$ and $\cos^2 x = 1 \Rightarrow x = 0, \pm\pi, \pm 2\pi$

Hence, total number of solutions is 5.

8. (390) $\pi < \alpha - \beta < 3\pi$

$$\Rightarrow \frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2} \Rightarrow \cos\frac{\alpha-\beta}{2} < 0$$

$$\sin\alpha + \sin\beta = -\frac{21}{65}$$

$$\Rightarrow 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -\frac{21}{65} \quad ....(1)$$

$$\cos\alpha + \cos\beta = -\frac{27}{65}$$

$$\Rightarrow 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -\frac{27}{65} \quad ....(2)$$

Square and add (1) and (2)

$$4\cos^2\frac{\alpha-\beta}{2} = \frac{(21)^2+(27)^2}{(65)^2} = \frac{1170}{65\times 65}$$

$$\therefore \cos^2\frac{\alpha-\beta}{2} = \frac{9}{130} \Rightarrow \cos\frac{\alpha-\beta}{2} = -\frac{3}{\sqrt{130}}$$

Chapter-4 : Principle of Mathematical Induction

1. (133) Putting $n = 1$ in $11^{n+2} + 12^{2n+1}$

We get, $11^{1+2} + 12^{2\times1+1} = 11^3 + 12^3 = 3059$, which is divisible by 133.

2. (11) Let $P(n) = 2 \cdot 4^{2n+1} + 3^{3n+1}$

Then $P(1) = 2 \cdot 4^3 + 3^4 = 209$, which is divisible by 11 but not divisible by 2, 7 or 27.

Further, let $P(k) = 2 \cdot 4^{2k+1} + 3^{3k+1}$ is divisible by 11, that is,

$2 \cdot 4^{2k+1} + 3^{3k+1} = 11q$ for some integer q.

Now $P(k+1) = 2 \cdot 4^{2k+3} + 3^{3k+4}$

$= 2 \cdot 4^{2k+1} \cdot 4^2 + 3^{3k+1} \cdot 3^3 = 16 \cdot 2 \cdot 4^{2k+1} + 27 \cdot 3^{3k+1}$

$= 16 \cdot 2 \cdot 4^{2k+1} + (16+11) \cdot 3^{3k+1}$

$= 16[2 \cdot 4^{2k+1} + 3^{3k+1}] + 11 \cdot 3^{3k+1}$

$= 16 \cdot 11q + 11 \cdot 3^{3k+1} = 11(16q + 3^{3k+1}) = 11m$

where $m = 16q + 3^{3k+1}$ is another integer.

$\therefore P(k+1)$ is divisible by 11.

$\therefore P(n) = 2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all $n \in N$.

3. (2) $2^4 \equiv 1 \pmod 5$; $\Rightarrow (2^4)^{75} \equiv (1)^{75} \pmod 5$

i.e. $2^{300} \equiv 1 \pmod 5 \Rightarrow 2^{300} \times 2 \equiv (1 \times 2) \pmod 5$

$\Rightarrow 2^{301} \equiv 2 \pmod 5$,

∴ Least positive remainder is 2.

4. (27) Let P(n) be the statement given by

P(n) : $41^n - 14^n$ is a multiple of 27

For n = 1,

i.e. $P(1) = 41^1 - 14^1 = 27 = 1 \times 27$,

which is a multiple of 27.

∴ P(1) is true.

Let P(k) be true, i.e. $41^k - 14^k = 27\lambda$ ... (i)

For n = k + 1,

$41^{k+1} - 14^{k+1} = 41^k\, 41 - 14^k\, 14$

$= (27\lambda + 14^k)\, 41 - 14^k\, 14$ [using (i)]

$= (27\lambda \times 41) + (14^k \times 41) - (14^k \times 14)$

$= (27\lambda \times 41) + 14^k (41 - 14) = (27\lambda \times 41) + (14^k \times 27)$

$= 27(41\lambda + 14^k)$,

which is a multiple of 27.

Therefore, P(k + 1) is true when P(k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

5. (9) $10^n + 3(4^{n+2}) + 5$

Taking $n = 2$;

$10^2 + 3 \times 4^4 + 5 = 100 + 768 + 5 = 873$

Therefore this is divisible by 9.

6. (6) $n(n^2 - 1) = (n-1)(n)(n+1)$

It is product of three consecutive natural numbers. So according to Langrange's theorem it is divisible by 3! i.e., 6

7. (24)

8 (25) Putting $n = 1$ in $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$

then, $7^{2\times1} + 2^{3\times1-3} \cdot 3^{1-1}$

$= 7^2 + 2^0 \cdot 3^0 = 49 + 1 = 50$ ...(i)

Also, $n = 2$

$7^{2\times2} + 2^{3\times2-3} \cdot 3^{2-1} = 2401 + 24 = 2425$ ...(ii)

From (i) and (ii) it is always divisible by 25.

Chapter-5 : Complex Numbers and Quadratic Equations

1. (8) Consider the equation

$x^2 + 2x + 2 = 0$

$$x = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

Let $\alpha=-1+i, \beta=-1-i$

$\alpha^{15}+\beta^{15}=(-1+i)^{15}+(-1-i)^{15}$

$$=\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^{15}+\left(\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^{15}$$

$$=\left(\sqrt{2}\right)^{15}\left[e^{\frac{i45\pi}{4}}+e^{\frac{-i45\pi}{4}}\right]$$

$$=\left(\sqrt{2}\right)^{15}.2\cos\frac{45\pi}{4}=-\left(\sqrt{2}\right)^{15}.2\cos\frac{\pi}{4}$$

$$=\frac{-2}{\sqrt{2}}\left(\sqrt{2}\right)^{15}$$

$$=-2\left(\sqrt{2}\right)^{14}=-256$$

2. **(4)** $\because z_0$ is a root of quadratic equation

$x^2+x+1=0$

$\therefore\quad z_0=\omega$ or $\omega^2 \rfloor z_0^3=1$

$\therefore\quad z=3+6i\,z_0^{81}-3i\,z_0^{93}$

$=3+6i((z_0)^3)^{27}-3i((z_0)^3)^{31}$

$=3+6i-3i$

$=3+3i$

$\therefore\quad \arg(z)=\tan^{-1}\left(\frac{3}{3}\right)=\frac{\pi}{4}$

3. **(0)** $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$$=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5+\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^5$$

$$=\left(e^{i\frac{\pi}{6}}\right)^5+\left(e^{-i\frac{\pi}{6}}\right)^5=-2\cos\frac{\pi}{6}=-\sqrt{3}$$

$\Rightarrow I(z)=0, R(z)=-\sqrt{3}$

4. (1.67) Since, $|z|+z=3+i$

Let $z=a+ib$, then

$|z|+z=3+i \Rightarrow \sqrt{a^2+b^2}+a+ib=3+i$

Compare real and imaginary coefficients on both sides

$b=1,\ \sqrt{a^2+b^2}+a=3$

$\sqrt{a^2+1}=3-a$

$a^2+1=a^2+9-6a$

$6a=8$

$a=\frac{4}{3}$

Then,

$$|z|=\sqrt{\left(\frac{4}{3}\right)^2+1}=\sqrt{\frac{16}{9}+1}=\frac{5}{3}$$

5. **(2)** Let $t=\frac{z-\alpha}{z+\alpha}$

$\because t$ is purely imaginary number.

$\therefore\quad t+\bar{t}=0$

$\Rightarrow\quad \frac{z-\alpha}{z+\alpha}+\frac{\bar{z}-\alpha}{\bar{z}+\alpha}=0$

$\Rightarrow\quad (z-\alpha)(\bar{z}+\alpha)+(\bar{z}-\alpha)(z+\alpha)=0$

$\Rightarrow\quad \bar{z}z-\alpha^2+z\bar{z}-\alpha^2=0$

$\Rightarrow\quad z\bar{z}-\alpha^2=0$

$\Rightarrow\quad |z|^2-\alpha^2=0$

$\Rightarrow\quad \alpha^2=4$

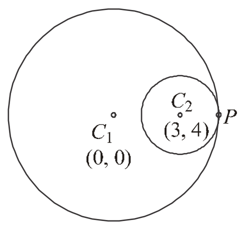
$\Rightarrow\quad \alpha=\pm 2$

6. **(0)** $|z_1|=9, |z_2-3-4i|=4$

z_1 lies on a circle with centre $C_1(0, 0)$ and radius $r_1=9$

z_2 lies on a circle with centre $C_2(3, 4)$ and radius $r_2=4$

So, minimum value of $|z_1-z_2|$ is zero at point of contact (i.e. A)



7. **(4)** The given quadratic equation is $x^2-2x+2=0$

Then, the roots of the this equation are $\frac{2\pm\sqrt{-4}}{2}=1\pm i$

Now, $\frac{\alpha}{\beta}=\frac{1-i}{1+i}=\frac{(1-i)^2}{1-i^2}=i$

or $\frac{\alpha}{\beta}=\frac{1-i}{1+i}=\frac{(1-i)^2}{1-i^2}=i$

So, $\frac{\alpha}{\beta}=\pm i$

Now, $\left(\frac{\alpha}{\beta}\right)^n=1\ (\pm i)^n=1$

n must be a multiple of 4.

Hence, the required least value of $n=4$.

8. **(1)** Let $z\in S$ then $z=\frac{\alpha+i}{\alpha-i}$

Since, z is a complex number and let $z=x+iy$

Then, $x+iy=\frac{(\alpha+i)^2}{\alpha^2+1}$ (by rationalisation)

$$\Rightarrow x+iy=\frac{(\alpha^2-1)}{\alpha^2+1}+\frac{i(2\alpha)}{\alpha^2+1}$$

Then compare both sides

$$x=\frac{\alpha^2-1}{\alpha^2+1} \quad ...(1)$$

$$y=\frac{2\alpha}{\alpha^2+1} \quad ...(2)$$

Now squaring and adding equations (1) and (2)

$$\Rightarrow x^2+y^2=\frac{(\alpha^2-1)^2}{(\alpha^2+1)^2}+\frac{4\alpha^2}{(\alpha^2+1)^2}=1$$

Chapter-6 : Linear Inequalities

1. (0) $2x-1=|x+7|=\begin{cases}x+7, \text{ if } x\ge -7\\ -(x+7), \text{ if } x<-7\end{cases}$

$\therefore$ If $x\ge -7$, $2x-1=x+7 \Rightarrow x=8$

If $x<-7$, $2x-1=-(x+7)$

$\Rightarrow 3x=-6$

$\Rightarrow x=-2$, which is not possible.

2. (4) Let x and x + 2 be two odd natural numbers.

we have, $x>10$ …(i)

and $x+(x+2)<40$ …(ii)

On solving (i) and (ii), we get

$10<x<19$

So, required pairs are (11, 13), (13, 15), (15, 17) and (17, 19)

3. (1) $x+\sqrt{3-x}\ge\sqrt{3-x}+3$

$\Rightarrow x\ge 3$

But $3-x\ge 0$

$\Rightarrow x\le 3$

Hence, $x=3$ is the only integral solution.

4. (1) $2^{x/2}+3^{x/2}=(\sqrt{13})^{x/2}$

$$\Rightarrow\left(\frac{2}{\sqrt{13}}\right)^{x/2}+\left(\frac{3}{\sqrt{13}}\right)^{x/2}=1$$

Which is of the form $\cos^{x/2}\alpha+\sin^{x/2}\alpha=1$.

$\therefore \frac{x}{2}=2.$

5. (2) $x=2+2^{2/3}+2^{1/3} \Rightarrow x-2=2^{2/3}+2^{1/3}$

Cubing both sides, we get

$x^3-8-6x^2+12x=6+6(x-2)$

$\Rightarrow x^3-6x^2+6x=2.$

6. (4) The given equation is

$$\log_7\left[\log_5\left(\sqrt{x+5}+\sqrt{x}\right)\right]=0$$

$$\Rightarrow \log_5(\sqrt{x+5}+\sqrt{x})=1$$

$$\Rightarrow \sqrt{x+5}+\sqrt{x}=5$$

$$\Rightarrow \sqrt{x+5}=5-\sqrt{x}$$

$$\Rightarrow x+5=25-10\sqrt{x}+x$$

$$\Rightarrow 10\sqrt{x}=20$$

$$\Rightarrow \sqrt{x}=2$$

$$\Rightarrow x=4$$

Hence, the solution is $x=4$.

7. (1.5) $5\{x\}=x+[x]$ and $[x]-\{x\}=\frac{1}{2}$

since $x=[x]+\{x\}\Rightarrow 4\{x\}=2[x]$ and $[x]-\{x\}=\frac{1}{2}$

after solving $[x]=1$ and $\{x\}=\frac{1}{2}$ $\therefore$ $x=1+\frac{1}{2}=\frac{3}{2}=1.5$

8. (1.5) $4^x-3^{x-\frac{1}{2}}=3^{x+\frac{1}{2}}-2^{2x-1}\Rightarrow 4^x+2^{2x-1}=3^{x+\frac{1}{2}}+3^{x-\frac{1}{2}}$

$$\Rightarrow 2^{2x}\left(1+\frac{1}{2}\right)=3^x\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)\Rightarrow 2^{2x}\left(\frac{3}{2}\right)=3^x\left(\frac{4}{\sqrt{3}}\right)$$

$$\Rightarrow 2^{2x-3}=3^{x-\frac{3}{2}}\Rightarrow 2^{2x-3}=3^{\frac{2x-3}{2}}\Rightarrow 2^{2x-3}=(\sqrt{3})^{2x-3},$$

which holds if $2x-3=0\Rightarrow x=\frac{3}{2}$. The solution is $\left\{\frac{3}{2}\right\}$.

Chapter-7 : Permutations and Combinations

1. (374) Number of numbers with one digit = 4 = 4

Number of numbers with two digits = 4 × 5 = 20

Number of numbers with three digits = 4 × 5 × 5 = 100

Number of numbers with four digits = 2 × 5 × 5 × 5 = 250

$\therefore$Total number of numbers = 4 + 20 + 100 + 250 = 374

2. (90) Domain and codomain = {1, 2, 3, ... 20}.

There are five multiple of 4 as 4, 8, 12, 16 and 20.

and there are 6 multiple of 3 as 3, 6, 9, 12, 15, 18.

Since, whenever k is multiple of 4 then $f(k)$ is multiple of 3 then total number of arrangement

$= {}^6C_5 \times 5! = 6!$

Remaining 15 elements can be arranged in 15! ways.

Since, for every input, there is an output

$\Rightarrow$ function $f(k)$ in onto

$\therefore$ Total number of arrangement = 15! . 6!

$\Rightarrow$ a × b = 15 × 6 = 90

3. (120) Collecting different labels of balls drawn = $10 \times 9 \times 8$

$\because$ arrangement is not required.

$\therefore$ the number of ways in which the balls can be chosen is,

$$\frac{10\times9\times8}{3!}=120$$

4. (12) ${}^mC_2 \times 2 = {}^mC_1 \cdot {}^2C_1 \times 2 + 84$

$m(m-1) = 4m + 84$

$m^2 - 5m - 84 = 0$

$m^2 - 12m - 7m - 84 = 0$

$m(m-12) + 7(m-12) = 0$

$m = 12, m = -7$

$\because m > 0$

$\therefore m = 12$

5. (86,400) At first we have to accommodate those 5 animals in cages which can not enter in 4 small cages, therefore number of ways are ${}^6P_5$. Now after accommodating 5 animals we left with 5 cages and 5 animals, therefore number of ways are 5!. Hence required number of ways $= {}^6P_5 \times 5! = 86400$.

6. (375) Required number of numbers $= 3 \times 5 \times 5 \times 5 = 375$

7. (9) We have $2^n - 2 = 510$

$\Rightarrow 2^n = 512$

$\Rightarrow n = 9$.

8. (12) Since the man can go in 4 ways and can back in 3 ways. Therefore total number of ways are $4 \times 3 = 12$ ways.

Chapter-8 : Binomial Theorem

1. (15) Consider the expression $\left(\frac{1-t^6}{1-t}\right)^3$

$= (1-t^6)^3(1-t)^{-3}$

$$= (1-3t^6+3t^{12}-t^{18})\left(1+3t+\frac{3\cdot4}{2!}t^2 + \frac{3\cdot4\cdot5}{3!}t^3+\frac{3\cdot4\cdot5\cdot6}{4!}t^4 + \ldots\infty\right)$$

Hence, the coefficient of $t^4 = 1\cdot\frac{3\cdot4\cdot5\cdot6}{4!}$

$$=\frac{3\times4\times5\times6}{4\times3\times2\times1}=15$$

2. (12.25) $(x+10)^{50}+(x-10)^{50}$

$= a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$

$\therefore a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$

$= 2({}^{50}C_0x^{50} + {}^{50}C_2x^{48}\cdot 10^2 + {}^{50}C_4x^{46}\cdot 10^4 + \ldots)$

$\therefore \quad a_0 = 2.{}^{50}C_{50}10^{50}$

$a_2 = 2.{}^{50}C_2.10^{48}$

$$\therefore \quad \frac{a_2}{a_0} = \frac{{}^{50}C_2\times10^{48}}{{}^{50}C_{50}10^{50}}$$

$$=\frac{50\times49}{2\times100}=\frac{49}{4}=12.25$$

3. (24) $(x+\sqrt{x^3-1})^6+(x-\sqrt{x^3-1})^6$

$= 2[{}^6C_0x^6 + {}^6C_2x^4(x^3-1) + {}^6C_4x^2(x^3-1)^2 + {}^6C_6(x^3-1)^3]$

$= 2[x^6 + 15x^7 - 15x^4 + 15x^8 - 30x^5 + 15x^2 + x^9 - 3x^6 + 3x^3 - 1]$

Hence, the sum of coefficients of even powers of $x = 2[1-15+15+15-3-1] = 24$

4. (10) $\therefore$ fourth term is equal to 200.

$$T_4 = {}^6C_3\left(\sqrt{x^{\left(\frac{1}{1+\log_{10}x}\right)}}\right)^3\left(x^{\frac{1}{12}}\right)^3 = 200$$

$$\Rightarrow 20x^{\frac{3}{2(1+\log_{10}x)}}.x^{\frac{1}{4}} = 200$$

$$x^{\frac{1}{4}+\frac{3}{2(1+\log_{10}x)}} = 10$$

Taking $\log_{10}$ on both sides and putting $\log_{10}x = t$

$\left(\frac{1}{4}+\frac{3}{2(1+t)}\right)t = 1 \Rightarrow t^2 + 3t - 4 = 0$

$\Rightarrow t^2 + 4t - t - 4 = 0 \Rightarrow t(t+4) - 1(t+4) = 0$

$\Rightarrow t = 1$ or $t = -4$

$\log_{10}x = 1 \Rightarrow x = 10$

or $\log_{10}x = -4 \Rightarrow x = 10^{-4}$

According to the question $x > 1$, $\therefore x = 10$.

5. (7) Given expression

$[x+(x^3-1)^{1/2}]^5 + [x-(x^3-1)^{1/2}]^5$

$\because \{[(x+y)^n + (x-y)^n] = 2[{}^nC_0x^ny^0 + {}^nC_2x^{n-2}y^2 + {}^nC_4x^{n-4}y^4 + \ldots]\}$

$= 2[x^5 + {}^5C_2x^3\{(x^3-1)^{1/2}\}^2 + {}^5C_4x\{(x^3-1)^{1/2}\}^4]$

$= 2[x^5 + 10x^3(x^3-1) + 5x(x^3-1)^2]$

$= 2[5x^7 + 10x^6 + x^5 - 10x^4 - 10x^3 + 5x]$,

which is a polynomial of degree 7.

6. (9) $27^{40} = 3^{120}$

$3^{119} = (4-1)^{119} = {}^{119}C_0 4^{119} - {}^{119}C_1 4^{118}$

$+ {}^{119}C_2 4^{117} - {}^{119}C_3 4^{116} + ... + (-1)$

$\therefore\ 3^{119} = 4k - 1$

$\therefore\ 3^{120} = 12k - 3 = 12(k-1) + 9$

$\therefore$ the required remainder is 9.

7. **(1)** $\because\ t_r = {}^nC_{r-1}\, p^{n-r+1}.q^{r-1}$ and $t_{r+1} = {}^nC_r\, p^{n-r} q^r$

Now, $t_r = t_{r+1} \Rightarrow {}^nC_{r-1}p^{n-r+1}.q^{r-1} = {}^nC_r\, p^{n-r}.q^r$

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!}.p^{n-r+1}.q^{r-1} = \frac{n!}{r!(n-r)!}.p^{n-r}.q^r$$

$$\Rightarrow \frac{r}{n-r+1}.p = q \Rightarrow \frac{p}{q} = \frac{n-r+1}{r}$$

$$\Rightarrow \frac{p+q}{q} = \frac{n+1}{r}$$ [Adding 1 to both sides]

$$\Rightarrow \frac{(n+1)q}{r(p+q)} = 1.$$

8. **(10)** $T_3 = {}^nC_2(x)^{n-2}\left(-\frac{1}{2x}\right)^2$ and $T_4 = {}^nC_3(x)^{n-3}\left(-\frac{1}{2x}\right)^3$

But according to the condition,

$$\frac{-n(n-1)\times 3\times 2\times 1\times 8}{n(n-1)(n-2)\times 2\times 1\times 4} = \frac{1}{2} \Rightarrow n = -10.$$

Chapter-9 : Sequence and Series

1. **(52)** $S = \sum_{i=1}^{30} a_i = \frac{30}{2}[2a_1 + 29d]$

$T = \sum_{i=1}^{15} a_{(2i-1)} = \frac{15}{2}[2a_1 + 28d]$

Since, $S - 2T = 75$

$\Rightarrow 30a_1 + 435d - 30a_1 - 420d = 75$

$\Rightarrow d = 5$

Also, $a_5 = 27 \Rightarrow a_1 + 4d = 27$

$\Rightarrow a_1 = 7$

Hence, $a_{10} = a_1 + 9d = 7 + 9\times 5 = 52$

2. (7820) $S = 1 + 6 + \frac{9(1^2+2^2+3^2)}{7} + \frac{12(1^2+2^2+3^2+4^2)}{9} + \frac{15(1^2+2^2+3^2+4^2+5^2)}{11} + K$

$$S = \frac{3\cdot(1)^2}{3} + \frac{6\cdot(1^2+2^2)}{5} + \frac{9\cdot(1^2+2^2+3^2)}{7} + \frac{12\cdot(1^2+2^2+3^2+4^2)}{9} + K$$

Now, n^{th} term of the series,

$$t_n = \frac{3n\cdot(1^2+2^2+K+n^2)}{(2n+1)}$$

$$\Rightarrow t_n = \frac{3n\cdot n(n+1)(2n+1)}{6(2n+1)} = \frac{n^3+n^2}{2}$$

$$\therefore S_n = \Sigma t_n = \frac{1}{2}\left\{\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \frac{n(n+1)}{4}\left(\frac{n(n+1)}{2} + \frac{2n+1}{3}\right)$$

Hence, sum of the series upto 15 terms is,

$$S_{15} = \frac{15\times 16}{4}\left\{\frac{15\cdot 16}{2} + \frac{31}{3}\right\}$$

$$= 60\times 120 + 60\times\frac{31}{3}$$

$= 7200 + 620$

$= 7820$

3. **(2)** $S_n = \left(\frac{1-q^{n+1}}{1-q}\right),\ T_n = \dfrac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\left(\frac{q+1}{2}\right)}$

$$\Rightarrow T_{100} = \frac{1-\left(\frac{q+1}{2}\right)^{101}}{1-\left(\frac{q+1}{2}\right)}$$

$$Sn = \frac{1}{1-q} - \frac{q^{n+1}}{1-q},\ T_{100} = \frac{2^{101}-(q+1)^{101}}{2^{100}(1-q)}$$

Now, ${}^{101}C_1 + {}^{101}C_2 S_1 + {}^{101}C_3 S_2 + \equiv + {}^{101}C_{101} S_{100}$

$$= \left(\frac{1}{1-q}\right)({}^{101}C_2 + \equiv + {}^{101}C_{101})$$

$$-\frac{1}{1-q}({}^{101}C_2 q^2 + {}^{101}C_3 q^3 + \equiv + {}^{101}C_{101} q^{101}) + 101$$

$$= \frac{1}{1-q}(2^{101} - 1 - 101) - \left(\frac{1}{1-q}\right)((1+q)^{101} - 1 - {}^{101}C_1 q) + 101$$

$$= \frac{1}{1-q}[2^{101} - 102 - (1+q)^{101} + 1 + 101q] + 101$$

$$= \frac{1}{1-q}[2^{101} - 101 + 101q - (1+q)^{101}] + 101$$

$$= \left(\frac{1}{1-q}\right)[2^{101} - (1+q)^{101}] = 2^{100}\, T_{100}$$

Hence, by comparison $\alpha^{100} = 2^{100}$

$\Rightarrow \alpha = 2$

4. (41) $\dfrac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\dfrac{p^2}{q^2}$

$\Rightarrow \dfrac{2a_1+(p-1)d}{2a_1+(q-1)d}=\dfrac{p}{q}$

$\dfrac{a_1+\left(\frac{p-1}{2}\right)d}{a_1+\left(\frac{q-1}{2}\right)d}=\dfrac{p}{q}$

For $\dfrac{a_6}{a_{21}}$, $p=11, q=41 \Rightarrow \dfrac{a_6}{a_{21}}=\dfrac{11}{41}$

$\Rightarrow \dfrac{11}{41}=\dfrac{11}{k} \Rightarrow k=41$

5. (512) $ar^4=2$

$a\times ar\times ar^2\times ar^3\times ar^4\times ar^5\times ar^6\times ar^7\times ar^8$
$=a^9r^{36}=(ar^4)^9=2^9=512$

6. (3) $\displaystyle\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$

$\displaystyle =i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11}-i\sin\frac{2k\pi}{11}\right)$

$\displaystyle =i\sum_{k=1}^{10}e^{-\frac{2k\pi}{11}i}=i\left\{\sum_{k=0}^{10}e^{-\frac{2k\pi}{11}i}-1\right\}$

$=i\left[1+e^{-\frac{2\pi}{11}i}+e^{-\frac{4\pi}{11}i}+....11\text{ terms}\right]-i$

$=i\left[\dfrac{1-\left(e^{-\frac{2\pi}{11}}\right)^{11}}{1-e^{-\frac{2\pi}{11}i}}\right]-i \quad =i\left[\dfrac{1-e^{-2\pi i}}{1-e^{-\frac{2\pi}{11}i}}\right]-i$

$=i\times 0-i$ $\qquad [\because e^{-2\pi i}=1]$

$=-i=i^3$

Then, for smallest natural number k, $i^k=i^3\Rightarrow k=3$

7. (2) The product is $P=2^{1/4}.2^{2/8}.2^{3/16}.........$

$=2^{1/4+2/8+3/16+.........\infty}$

Now let $S=\dfrac{1}{4}+\dfrac{2}{8}+\dfrac{3}{16}+........\infty$ ......(1)

$\dfrac{1}{2}S=\dfrac{1}{8}+\dfrac{2}{16}+........\infty$ ......(2)

Subtracting (2) from (1)

$\Rightarrow \quad \dfrac{1}{2}S=\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+........\infty$

or $\quad \dfrac{1}{2}S=\dfrac{1/4}{1-1/2}=\dfrac{1}{2}\Rightarrow S=1$

$\therefore P=2^S=2$

8. (11) Let, $S=\displaystyle\sum_{k=1}^{20}k.\frac{1}{2^k}$

$S=\dfrac{1}{2}+2.\dfrac{1}{2^2}+3.\dfrac{1}{2^3}+....+20.\dfrac{1}{2^{20}}$ ...(i)

$\dfrac{1}{2}S=\dfrac{1}{2^2}+2.\dfrac{1}{2^3}+....+19\dfrac{1}{2^{20}}+20\dfrac{1}{2^{21}}$ ...(ii)

On subtracting equations (ii) by (i),

$=\dfrac{S}{2}=\left(\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+....+\dfrac{1}{2^{20}}\right)-20\dfrac{1}{2^{21}}$

$=\dfrac{\frac{1}{2}\left(1-\frac{1}{2^{20}}\right)}{1-\frac{1}{2}}-20.\dfrac{1}{2^{21}}=1-\dfrac{1}{2^{20}}-10.\dfrac{1}{2^{20}}$

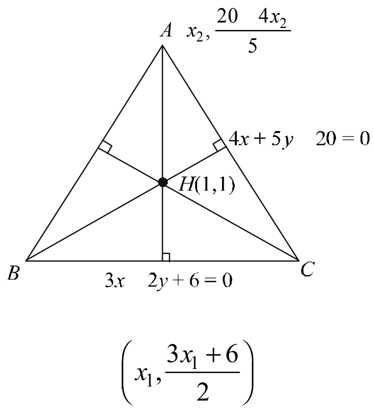
$\dfrac{S}{2}=1-11.\dfrac{1}{2^{20}}\Rightarrow S=2-11.\dfrac{1}{2^{19}}=2-\dfrac{11}{2^{19}}$

$\Rightarrow 2-\dfrac{k}{2^{19}}=2-\dfrac{11}{2^{19}}$

$\Rightarrow -k=-11$
$\Rightarrow k=11$

Chapter-10 : Straight Lines

1. (1579)



Since, AH is perpendicular to BC

Hence, $m_{AH}\,.\,m_{BC}=-1$

$\left(\dfrac{\frac{20-4x_2}{5}-1}{x_2-1}\right)\times\dfrac{3}{2}=-1$

$\dfrac{15-4x_2}{5(x_2-1)}=-\dfrac{2}{3}$

$45-12x_2=-10x_2+10$

$2x_2 = 35 \Rightarrow x_2 = \frac{35}{2}$

$\Rightarrow A\left(\frac{35}{2}, -10\right)$

Since, BH is perpendicular to CA.

Hence, $m_{BH} \times m_{CA} = -1$

$$\left(\frac{\frac{3x_1}{2}+3-1}{x_1-1}\right)\left(-\frac{4}{5}\right) = -1$$

$$\frac{(3x_1+4)}{2(x_1-1)} \times 4 = 5$$

$\Rightarrow 6x_1 + 8 = 5x_1 - 5 \Rightarrow x_1 = -13 \Rightarrow \left(-13, \frac{-33}{2}\right)$

$\Rightarrow$ Equation of line AB is

$$y + 10 = \left(\frac{-\frac{33}{2}+10}{-13-35}\right)\left(x - \frac{35}{2}\right)$$

$\Rightarrow -61y - 610 = -13x + \frac{455}{2}$

$\Rightarrow -122y - 1220 = -26x + 455$

$\Rightarrow 26x - 122y - 1675 = 0$

Equation of third side is $26x - 122y - 1675 = 0$

$\Rightarrow ax + by + c = 26x - 122y - 1675$

$a = 26, b = -122, c = -1675$

$\therefore a + b - c = 26 + (-122) - (-1675)$

$= 1579$

2. **(5)** $\because$ In a rectangle $ABCD$, the diagonals are equal

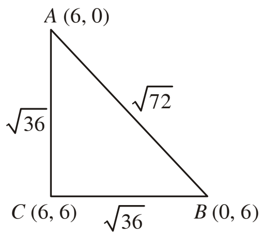
Therefore, $AC = BD$

Now length of $BD = \sqrt{3^2 + 4^2} = 5$

$\Rightarrow AC = 5$.

3. **(1.41)** Centroid of the triangle is,

$$G = \left(\frac{6+0+6}{3}, \frac{0+6+6}{3}\right)$$



$\therefore$ centroid $G(4, 4)$

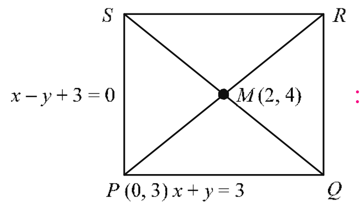
By calculating length of AB, BC and AC, we find that, ΔABC is right angled triangle

$\therefore$ circumcentre is mid point of hypotenuse $= (3, 3)$

Hence, distance between centroid (4, 4) and circumcentre (3, 3) is,

$= \sqrt{(4-3)^2 + (4-3)^2} = \sqrt{2} = 1.41$

4. **(9)**



Since, $x - y + 3 = 0$ and $x + y = 3$ are perpendicular lines and intersection point of $x - y + 3 = 0$ and $x + y = 3$ is $P(0, 3)$.

$\Rightarrow M$ is mid-point of $PR \Rightarrow R(4, 5)$

Let $S(x_1, x_1 + 3)$ and $Q(x_2, 3 - x_2)$

M is mid-point of SQ

$\Rightarrow x_1 + x_2 = 4, x_1 + 3 + 3 - x_2 = 8$

$\Rightarrow x_1 = 3, x_2 = 1$

Then, the vertex D is (3, 6).

$\Rightarrow (a, b) = (3, 6) \Rightarrow a = 3, b = 6$

$\therefore a + b = 3 + 6 = 9$

5. **(2)** Equation of bisectors of second pair of straight lines is,

$qx^2 + 2xy - qy^2 = 0$ ....(1)

It must be identical to the first pair

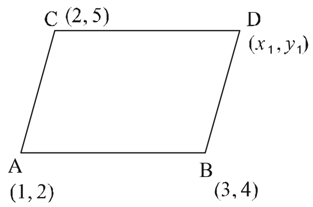
$x^2 - 2pxy - y^2 = 0$ ....(2)

from (1) and (2)

$\frac{q}{1} = \frac{2}{-2p} = \frac{-q}{-1} \Rightarrow pq = -1 \Rightarrow pq + 3 = -1 + 3 = 2$

6. **(3)** Since, in parallelogram mid points of both diagonals coinsides.

$\therefore$ $\therefore$ mid-point of AD = mid-point of BC



$$\left(\frac{x_1+1}{2}, \frac{y_1+2}{2}\right) = \left(\frac{3+2}{2}, \frac{4+5}{2}\right)$$

$\therefore (x_1, y_1) = (4, 7)$

Then, equation of AD is,

$$y - 7 = \frac{2-7}{1-4}(x-4)$$

$$y - 7 = \frac{5}{3}(x-4)$$

$3y - 21 = 5x - 20$

$5x - 3y + 1 = 0$

$\because$ Equation of diagonal AD is $ax + by + c = 0$

Then $ax + by + c = 5x - 3y + 1$

$\Rightarrow a = 5, b = -3, c = 1$

Hence, $a + b + c = 5 - 3 + 1 = 3$

7. **(0)** $3x + 4y = 0$ is one of the lines of the pair

$6x^2 - xy + 4cy^2 = 0$, Put $y = -\frac{3}{4}x$,

we get $6x^2 + \frac{3}{4}x^2 + 4c\left(-\frac{3}{4}x\right)^2 = 0$

$\Rightarrow 6 + \frac{3}{4} + \frac{9c}{4} = 0 \Rightarrow c = -3 \Rightarrow c + 3 = 0$

8. **(4)** $\because (h, k), (1, 2)$ and $(-3, 4)$ are collinear

$\therefore \begin{vmatrix} h & k & 1 \\ 1 & 2 & 1 \\ -3 & 4 & 1 \end{vmatrix} = 0 \Rightarrow -2h - 4k + 10 = 0$

$\Rightarrow h + 2k = 5$ ...(i)

Now, $m_{L_1} = \frac{4-2}{-3-1} = -\frac{1}{2} \Rightarrow m_{L_2} = 2$ $[\because L_1 \perp L_2]$

By the given points (h, k) and $(4, 3)$,

$m_{L_2} = \frac{k-3}{h-4} \Rightarrow \frac{k-3}{h-4} = 2 \Rightarrow k - 3 = 2h - 8$

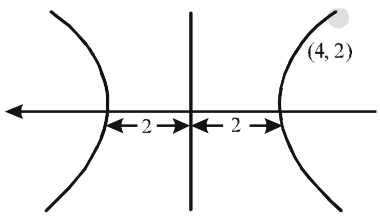
$2h - k = 5$ ...(ii)

From (i) and (ii)

$k + h = 1 + 3 = 4$

Chapter-11 : Conic Sections

1. **(3)**



Consider equation of hyperbola

$$\frac{x^2}{2^2} - \frac{y^2}{b^2} = 1$$

$\because$ (4, 2) lies on hyperbola

$\therefore \frac{16}{4} - \frac{4}{b^2} = 1$

$\therefore b^2 = \frac{4}{3}$

Since, eccentricity $= \sqrt{1 + \frac{b^2}{a^2}}$

Hence, eccentricity $= \sqrt{1 + \frac{\frac{4}{3}}{4}} = \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}}$

$\therefore \frac{2}{\sqrt{a}} = \frac{2}{\sqrt{3}} \Rightarrow a = 3$

2. **(5)** Given straight line,

$5x - 2y + 6 = 0$ ...(i)

For the point of intersection of the line (i) with y-axis, we put $x = 0$

we get $y = 3$

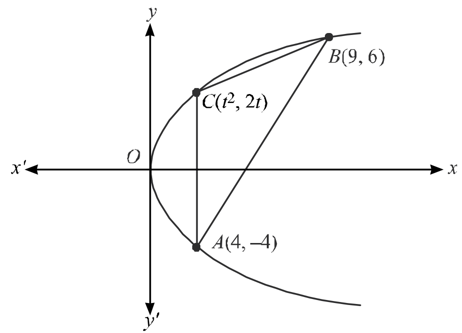
Hence, $Q = (0, 3)$

Given eqn of circle is

$x^2 + y^2 + 6x + 6y - 2 = 0$ ...(ii)

$PQ = \sqrt{0 + (3)^2 + 6 \times 0 + 6 \times 3 - 2} = \sqrt{9 + 16} = 5$

3. (31.25)



Let the coordinates of C is $(t^2, 2t)$.

Since, area of ΔACB

$= \frac{1}{2}\left\|\begin{matrix} t^2 & 2t & 1 \\ 9 & 6 & 1 \\ 4 & -4 & 1 \end{matrix}\right\|$

$= \frac{1}{2}|t^2(6+4) - 2t(9-4) + 1(-36-24)|$

$= \frac{1}{2}|10t^2 - 10t - 60|$

$= 5|t^2 - t - 6|$

$= 5\left|\left(t - \frac{1}{2}\right)^2 - \frac{25}{4}\right|$ [Here, t (0, 3)]

For maximum area, $t = \frac{1}{2}$

Hence, maximum area $= \frac{125}{4} = 31.25$ sq. units

4. **(25)** Two circles $x^2 + y^2 - 2x - 2y - 7 = 0$ ...(i)

and $x^2 + y^2 + 4x + 2y + k = 0$ ...(ii)

$\because$ circles cut orthogonally,

$\therefore 2(g_1g_2 + f_1f_2) = c_1 + c_2$

$2((-2) + (-1)) = -7 + k \Rightarrow 2(-3) = -7 + k$

$\Rightarrow -6 + 7 = k \Rightarrow k = 1$

Now, equation of common chord is

$S_1 - S_2 = (x^2 + y^2 - 2x - 2y - 7)$
$\qquad - (x^2 + y^2 + 4x + 2y + 1) = 0$

$\Rightarrow -6x - 4y - 8 = 0 \Rightarrow 6x + 4y + 8 = 0$

$\Rightarrow 3x + 2y + 4 = 0$ ...(iii)

Length of perpendicular from the centre C_2 (–2, –1) of circle (ii) upon the common chord (iii) is

$$C_2M = \frac{3(-2)+2(-1)+4}{\sqrt{(3)^2+(2)^2}} = \frac{-6-2+4}{\sqrt{9+4}} = \frac{-4}{\sqrt{13}}$$

Radius of circle (ii),

$C_2P = \sqrt{4+1-1} = \sqrt{4} = 2$

So, the length of the common chord is given by

$PQ = 2PM = 2\sqrt{(C_2P)^2-(C_2M)^2}$

$$= 2\sqrt{4-\frac{16}{13}} = 2\sqrt{\frac{52-16}{13}} = \frac{12}{\sqrt{13}}.$$

Now, $\frac{m}{\sqrt{n}} = \frac{12}{\sqrt{13}} \Rightarrow m = 12$ and $n = 13$

Hence, $m+n = 12+13 = 25$

5. **(6)** Let intersection points be $P(x_1, y_1)$ and $Q(x_2, y_2)$

The given equations

$x^2 = 4y$ ...(1)

$x - \sqrt{2}y + 4\sqrt{2} = 0$ ...(2)

Use eqn. (1) in eqn. (2)

$$x - \sqrt{2}\frac{x^2}{4} + 4\sqrt{2} = 0$$

$$\sqrt{2}x^2 - 4x - 16\sqrt{2} = 0$$

$x_1 + x_2 = 2\sqrt{2}, x_1x_2 = -16, (x_1 - x_2)^2 = 8 + 64 = 72$

Since, points P and Q both satisfy the equations (2), then

$x_1 - \sqrt{2}y_1 + 4\sqrt{2} = 0$

$x_1 - \sqrt{2}y_2 + 4\sqrt{2} = 0$

$(x_2 - x_1) = \sqrt{2}(y_2 - y_1) \Rightarrow (x_2 - x_1)^2 = 2(y_2 - y_1)^2$

$PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$$= \sqrt{(x_2-x_1)^2 + \frac{(x_2-y_1)^2}{2}}$$

$$= |x_2 - x_1| \cdot \frac{\sqrt{3}}{\sqrt{2}} = 6\sqrt{2} \times \frac{\sqrt{3}}{\sqrt{2}} = 6\sqrt{3}$$

Then, length of chord $= 6\sqrt{3}$.

$\therefore a\sqrt{3} = 6\sqrt{3} \Rightarrow a = 6$

6. **(5)** Clearly the point (1, 2) is the centre of the given circle and infinite tangents can only be drawn on a point circle. Hence the radius should be 0.

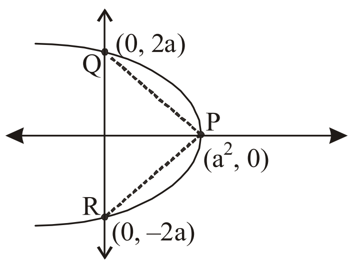
$\therefore \sqrt{1^2 + 2^2 - \lambda} = 0 \Rightarrow \lambda = 5$

7. **(8)** $a = \left|\frac{-8}{\sqrt{1+1}}\right| - \left|\frac{-12}{\sqrt{1+1}}\right| = \frac{4}{\sqrt{2}}$

Length of latus rectum $= 4a = 4 \times \frac{4}{\sqrt{2}} = 8\sqrt{2}$.

$\Rightarrow a\sqrt{2} = 8\sqrt{2} \Rightarrow a = 8$

8. **(5)** $y^2 = -4(x - a^2)$



$$\text{Area} = \frac{1}{2}(4a)(a^2)$$

$= 2a^3$

Since $2a^3 = 250$

$\Rightarrow a = 5$

Chapter-13 : Limits and Derivatives

1. **(1)** $\lim\limits_{x\to 0} \frac{x\cot 4x}{\sin^2 x.\cot^2 2x}$

$$= \lim_{x\to 0} \frac{x.\tan^2 2x}{\sin^2 x.\tan 4x}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 \cdot \left(\frac{\tan 2x}{2x}\right)^2 \cdot \left(\frac{4x}{\tan 4x}\right) \cdot \frac{4}{2^2} = 1$$

2. **(8)** $\lim\limits_{x\to\frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = \lim\limits_{x\to\frac{\pi}{4}} \frac{\cot^3 x\left(1 - \frac{\tan x}{\cot^3 x}\right)}{\cos(x + \pi/4)}$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(1-\tan^4 x)}{\tan^3 x\cos(x+\pi/4)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(1+\tan^2 x)(1-\tan x)(1+\tan x)}{\tan^3 x\left(\frac{\cos x - \sin x}{\sqrt{2}}\right)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{(1+\tan^2 x)(1+\tan x)(\cos x - \sin x)}{\frac{\sin^3 x}{\cos^2 x}\left(\frac{\cos x - \sin x}{\sqrt{2}}\right)}$$

$$= \frac{(2)(2)}{\frac{1}{(\sqrt{2})(\sqrt{2})}} = 8$$

3. **(4)** $\lim\limits_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$

$$= \lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{2\cos^2 \frac{x}{2}}} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}\left[1-\cos\frac{x}{2}\right]} = \lim_{x\to 0}\frac{\sin^2 x}{2\sqrt{2}\sin^2\frac{x}{4}}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin x}{x}\right)^2.16}{2\sqrt{2}\left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^2} = \frac{16}{2\sqrt{2}} = 4\sqrt{2}$$

$\Rightarrow a\sqrt{2} = 4\sqrt{2} \quad \Rightarrow a = 4$

4. **(0.5)** $\lim_{x\to 0}\frac{\tan 2x - x}{3x - \sin x} = \lim_{x\to 0}\left\{\frac{\frac{2\tan 2x}{2x}-1}{3-\frac{\sin x}{x}}\right\} = \frac{2-1}{3-1} = \frac{1}{2} = 0.5$

5. **(3)** Let $L = \lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$ $\left[\frac{0}{0}\text{form}\right]$

$\therefore$ Applying L Hopital's rule, we get

$$L = \lim_{h\to 0}\frac{f'(2h+2+h^2).(2+2h)}{f'(h-h^2+1).(1-2h)}$$

$$= \frac{f'(2).2}{f'(1).1} = \frac{6\times 2}{4\times 1} = 3.$$

6. **(1)** $\lim_{x\to\pi/2}\left[x\tan x - \left(\frac{\pi}{2}\right)\sec x\right]$

$$= \lim_{x\to\frac{\pi}{2}}\left[\frac{x\sin x}{\cos x} - \frac{\pi}{2\cos x}\right]$$

$$= \lim_{x\to\pi/2}\frac{2x\sin x - \pi}{2\cos x}$$

$$= \lim_{x\to\pi/2}\frac{[2\sin x + 2x\cos x]}{-2\sin x} = -1$$

$\Rightarrow -a = -1$

$\Rightarrow a = 1$ [By L-Hospital's rule]

7. **(0)** $f(x) = \sqrt{x-1}+\sqrt{25+(x-1)-10\sqrt{x-1}}$

$= \sqrt{x-1}+\sqrt{(5-\sqrt{x-1})^2} = \sqrt{x-1}+|5-\sqrt{x-1}| = 5$

$[\because \sqrt{x-1} < 5 \text{ for } 1 < x < 26]$

$\therefore \quad f'(x) = 0$

8. **(0)** Given : $f(x) = |x-1|+|x-3|$

At $x = 2, |x-1| = x-1$

and $|x-3| = -x+3 \Rightarrow f(x) = x-1-x+3 = 2$

which is a constant function $\Rightarrow f'(2) = 0$

Chapter-15 : Statistics

1. **(20)** Variance $= \sigma^2 = \frac{\Sigma x_i^2}{N} - (\bar{x})^2$

$\Rightarrow 18 = \frac{\Sigma x_i^2}{5} - (150)^2$

$\Rightarrow \Sigma x_i^2 = 90 + 112590 = 112590$

Then, variance of the height of six students

$$V_2 = \frac{112590+(156)^2}{6} - \left(\frac{750+156}{6}\right)^2$$

$= 22821 - 22801$

$= 20$

2. **(5)** Variance is given by,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

$$\sigma^2 = \frac{1}{n}A - \frac{1}{n^2}B^2 \quad ...(i)$$

Here, $A \sum_{i=1}^{n}x_i^2 =$ and $B = \sum_{i=1}^{n}x_i$

$\because \sum_{i=1}^{n}(x_i+1)^2 = 9n$

$\Rightarrow A + n + 2B = 9n \rfloor A + 2B = 8n$ ...(ii)

$\because \sum_{i=1}^{n}(x_i-1)^2 = 5n$

$\Rightarrow A + n - 2B = 5n \rfloor A - 2B = 4n$ ...(iii)

From (ii) and (iii),

$A = 6n, B = n$

$\Rightarrow \sigma^2 = \frac{1}{n}\times 6n - \frac{1}{n^2}\times n^2 = 6-1 = 5$

$\Rightarrow \sigma = \sqrt{5}$

$\Rightarrow \sqrt{a} = \sqrt{5} \Rightarrow a = 5$

3. **(507.5)** $\because \bar{x} = \frac{\sum_{i=1}^{5}x_i}{5} \Rightarrow \sum_{i=1}^{5}x_i = 10\times 5 = 50$

$\Rightarrow \sum_{i=1}^{6}xi = 50 - 50 = 0$

$\frac{\sum_{i=1}^{5}x_i^2}{5} - (10)^2 = 3^2 = 9$

$\Rightarrow \sum_{i=1}^{5}x_i^2 = 545$

Then,

$\Rightarrow \sum_{i=1}^{6}x_i^2 = \sum_{i=1}^{5}x_i^2 + (-50)^2$

$= 545 + (-50)^2 = 3045$

$$\text{Variance} = \frac{\sum_{i=1}^{6} x_i^2}{6} - \left(\frac{\sum_{i=1}^{6} x_i}{6}\right)^2 = \frac{3045}{6} - 0 = 507.5$$

4. **(65)** Total student = 100;
for 70 students total marks = 75 × 70 = 5250
⇒ Total marks of girls = 7200 – 5250
= 1950

$\text{Average of girls} = \frac{1950}{30} = 65$

5. **(24)** Mode + 2Mean = 3 Median
⇒ Mode + 2 × 21 = 3 × 22
⇒ Mode = 66 – 42 = 24

6. **(1)** $\sigma_x^2 = \frac{\sum d_i^2}{n}$ (Here deviations are taken from the mean). Since A and B both have 100 consecutive integers, therefore both have same standard deviation and hence the variance. $\therefore \frac{V_A}{V_B} = 1$
(As $\sum d_i^2$ is same in both the cases)

7. **(78)** $\sum x = 170, \sum x^2 = 2830$ increase in $\sum x = 10$, then
$\sum x' = 170 + 10 = 180$
Increase in $\sum x^2 = 900 - 400 = 500$ then
$\sum x'^2 = 2830 + 500 = 3330$

$$\text{Variance} = \frac{1}{n}\sum x'^2 - \left(\frac{1}{n}\sum x'\right)^2$$

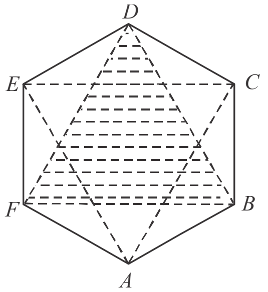
$$= \frac{1}{15} \times 3330 - \left(\frac{1}{15} \times 180\right)^2 = 222 - 144 = 78.$$

8. **(15)** $\because$ Q.D. $= \frac{5}{6} \times M.D. = \frac{5}{6} \times 12 = 10$

Also S.D. $= \frac{3}{2} \times Q.D. = \frac{3}{2} \times 10 \Rightarrow S.D. = 15.$

Chapter-16: Probability – I

1. (0.1) Three vertices out of 6 can be chosen in ${}^6C_3$ ways.
So, total ways = ${}^6C_3 = 20$



Only two equilateral triangles can be formed $\Delta\, AEC$ and $\Delta\, BFD$.
So, favourable ways = 2.

$\therefore$ required probability $= \frac{2}{20} = \frac{1}{10} = 0.1$

2. (0.8) Here, two numbers are selected from {1, 2, 3, 4, 5, 6}
⇒ $n(S) = 6 \times 5$ (without replacement)
$n(E) = 6 \times 4$ (as for the minimum of the two is not more than 4)

$\therefore$ required probability $= \frac{n(E)}{n(S)} = \frac{24}{30} = \frac{4}{5} = 0.8$

3. (0.8) Total no. of arrangements of the letters of the word UNIVERSITY is $\frac{10!}{2!}$.
No. of arrangements when both I's are together = 9!
So. the no. of ways in which 2 I's do not together

$= \frac{10!}{2!} - 9!$

$\therefore$ Required probability

$$= \frac{\frac{10!}{2!} - 9!}{\frac{10!}{2!}} = \frac{10! - 9!\,2!}{10!}$$

$$= \frac{10 \times 9! - 9!.2!}{10!} = \frac{9![10-2]}{10 \times 9!}$$

$$= \frac{8}{10} = \frac{4}{5} = 0.8$$

4. **(3)** Let each of the friend have x daughters. Then the probability that all the tickets go to daughters of A is $\frac{{}^xC_3}{{}^{2x}C_3}$.

$\therefore \frac{{}^xC_3}{{}^{2x}C_3} = \frac{1}{20} \Rightarrow x = 3$

5. **(1)** $n(A) = \{4, 5, 6\}, n(B) = \{1, 2, 3, 4\}$ and $n(A \cap B) = \{4\}$
$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = 1$

6. (0.40) Let 5 horses are H_1, H_2, H_3, H_4 and H_5. Selected pair of horses will be one of the 10 pairs (i.e.; ${}^5C_2$): $H_1 H_2$, $H_1 H_3$, $H_1 H_4$, $H_1 H_5$, H_2H_3, $H_2 H_4$, $H_2 H_5$, $H_3 H_4$, $H_3 H_5$ and $H_4 H_5$.
Any horse can win the race in 4 ways.
For example : Horses H_2 win the race in 4 ways $H_1 H_2$, H_2H_3, H_2H_4 and H_2H_5.

Hence required probability $= \frac{4}{10} = \frac{2}{5} = 0.40$

7. (60) $P(A \cup B) = P(A) + P(B) - P(A \cap B);$

$\Rightarrow \frac{3}{4} = 1 - P(\overline{A}) + P(B) - \frac{1}{4}$

$\Rightarrow 1 = 1 - \frac{2}{3} + P(B) \Rightarrow P(B) = \frac{2}{3};$

Now, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12} \Rightarrow \frac{5}{12} = \frac{a}{b} \Rightarrow a = 5, b = 12$

$\therefore a \times b = 5 \times 12 = 60$

8. (55000) Out of 1000 pages, ${}^{9+3-1}C_{3-1} = 55$ pages are such that sum of digits of page number is 9.

$\therefore$ required probability $= \frac{55}{1000}$

Chapter-17 : Relations and Functions – II

1. **(14)** If set A has m elements and set B has n elements then number of onto functions from A to B is

$$\sum_{r=1}^{n} (-1)^{n-r}\ {}^{n}C_r\, r^m \text{ where } 1 \le n \le m$$

Here $E = \{1,2,3,4\}$, $F = \{1,2\}$; $m = 4, n = 2$

$\therefore$ no. of onto functions from E to F

$$= \sum_{r=1}^{2} (-1)^{2-r}\ {}^{2}C_r (r)^4 = (-1)\ {}^{2}C_1 + {}^{2}C_2 (2)^4$$

$= -2 + 16 = 14$

2. **(7)** R is reflexive if it contains $(1,1),(2,2),(3,3)$

$\because (1,2) \in R, (2,3) \in R$

$\because$ R is symmetric if $(2,1),(3,2) \in R.$

Now,

$R = \{(1,1),(2,2),(3,3),(2,1),(3,2),(2,3),(1,2)\}$

R will be transitive if $(3,1),\ (1,3) \in R.$

Thus, R becomes an equivalence relation by adding (1, 1) (2, 2) (3, 3) (2, 1) (3, 2) (1, 3) (3,1).

Hence, the total number of ordered pairs is 7.

3. **(24)** The total number of injective functions from a set A containing 3 elements to a set B containing 4 elements is equal to the total number of arrangement of 4 by taking 3 at a time i.e., ${}^{4}P_3 = 24.$

4. **(1)** $\because$ $\text{fog}\left(-\frac{1}{4}\right) = \text{f}\left[\text{g}\left(-\frac{1}{4}\right)\right] = \text{f}(-1) = 1$

and $\text{gof}\left(-\frac{1}{4}\right) = \text{g}\left[\text{f}\left(-\frac{1}{4}\right)\right] = \text{g}\left(\frac{1}{4}\right) = [1/4] = 0$

$\therefore$ required value $= 1 + 0 = 1$

5. **(1)** Given that,

$$f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left[2\sin^2 x + 2\sin^2\left(x + \frac{\pi}{3}\right) + 2\cos x \cos\left(x + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[1 - \cos 2x + 1 - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[1 + 1 + \frac{1}{2} - \cos 2x + \cos\left(2x + \frac{\pi}{3}\right) - \cos\left(2x + \frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + 2\cos 2x \times \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + \cos 2x\right] = \frac{5}{4}$$

Now, $(gof)(x) = g(f(x)) = g\left(\frac{5}{4}\right) = 1$

6. **(23)** $\because$ $\phi(x) = ((hof)og)(x)$

$\therefore \phi\left(\frac{\pi}{3}\right) = h\left(f\left(g\left(\frac{\pi}{3}\right)\right)\right) = h(f(\sqrt{3})) = h(3^{1/4})$

$= \frac{1-\sqrt{3}}{1+\sqrt{3}} = -\frac{1}{2}(1 + 3 - 2\sqrt{3}) = \sqrt{3} - 2 = -(-\sqrt{3} + 2)$

$= -\tan 15° = \tan(180° - 15°) = \tan\left(\pi - \frac{\pi}{12}\right) = \tan\frac{11\pi}{12}$

Now, $\tan\frac{p\pi}{q} = \tan\frac{11\pi}{12}$

$\Rightarrow p = 11, q = 12$

Hence, $p + q = 11 + 12 = 23$

7. **(1)** $f(x) = \frac{\alpha x}{x+1}, x \ne -1$ (given)

$$f[f(x)] = f\left(\frac{\alpha x}{x+1}\right) = \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1}$$

$$= \frac{\frac{\alpha^2 x}{x+1}}{\frac{\alpha x + (x+1)}{x+1}} = \frac{\alpha^2 x}{(\alpha+1)x + 1} = x \quad \text{(given)}$$

$(\alpha + 1)x^2 + (1 - \alpha^2)x = 0$

$\Rightarrow \alpha + 1 = 0\ \&\ (1 - \alpha^2) = 0$

$\Rightarrow \alpha = -1\ \&\ \alpha = \pm 1$

$\because \alpha > 0$

$\therefore \alpha = 1$

8. **(8)** Given $f(x) = x^2 + 1$

Let $f^{-1}(17) = x \Rightarrow f(x) = 17$

$\Rightarrow x^2 + 1 = 17 \Rightarrow x = \pm 4$

Similarly, $f^{-1}(-3) = x$ gives

$x^2 + 1 = -3 \Rightarrow x^2 = -4$ (not possible)

$\therefore \quad x = +4$ or -4

$\therefore |f^{-1}(17)| + |f^{-1}(-3)| = 4 + 4 = 8$

Chapter-18 : Inverse Trigonometric Functions

1. (40) $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$

$= \cot\left(\sum_{n=1}^{19}\cot^{-1}(1+n(n+1))\right)$

$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$

$\left[\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) : \text{for } x>0\right]$

$= \cot\left(\sum_{n=1}^{19}(\tan^{-1}(n+1) - \tan^{-1}n)\right)$

$= \cot(\tan^{-1}20 - \tan^{-1}1)$

$= \cot\left(\tan^{-1}\left(\frac{20-1}{1+20\times 1}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{19}{21}\right)\right) = \cot\cot^{-1}\left(\frac{21}{19}\right) = \frac{21}{19}$

$\because \frac{m}{n} = \frac{21}{19} \Rightarrow m = 21, n = 19$

$\therefore m + n = 21 + 19 = 40$

2. (210) Let $\tan^{-1}\frac{1}{3} = \alpha$ and $\tan^{-1}2\sqrt{2} = \beta$

$\therefore \sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1}2\sqrt{2}) = \sin 2\alpha + \cos\beta$

$= \frac{2\tan\alpha}{1+\tan^2\alpha} + \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{2\times 1/3}{1+1/9} + \frac{1}{\sqrt{1+8}} = \frac{14}{15}$

$\frac{p}{q} = \frac{14}{15} \Rightarrow p = 14, q = 15; \therefore pq = 14 \times 15 = 210$

3. (133) $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}; \left(x > \frac{3}{4}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right)$

$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \sin^{-1}\left(\frac{3}{4x}\right)$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

Put $\Rightarrow \sin^{-1}\left(\frac{3}{4x}\right) = \theta \Rightarrow \sin\theta = \frac{3}{4x}$

$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{16x^2}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$

$\therefore \quad \cos^{-1}\left(\frac{2}{3x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$

$\Rightarrow \frac{2}{3x} = \frac{\sqrt{16x^2-9}}{4x} \Rightarrow x^2 = \frac{64+81}{9\times 16} \Rightarrow x = \pm\sqrt{\frac{145}{144}}$

$\Rightarrow x = \frac{\sqrt{145}}{12} \Rightarrow \frac{\sqrt{p}}{q} = \frac{\sqrt{145}}{12} \qquad \left(\because x > \frac{3}{4}\right)$

$\Rightarrow p = 145, q = 12$

Then, $p - q = 145 - 12 = 133$

4. (2) We have $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ and

$\sin^{-1}x \le \frac{\pi}{2}$ it is possible only when

$\sin^{-1}x = \frac{\pi}{2} \Rightarrow x = 1$

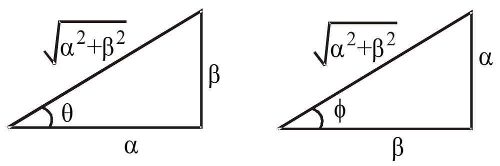
$\sin^{-1}y = \frac{\pi}{2} \Rightarrow y = 1; \qquad \sin^{-1}z = \frac{\pi}{2} \Rightarrow z = 1$

$\therefore x^{100} + y^{100} + z^{100} - \frac{3}{x^{101}+y^{101}+z^{101}}$

$= 1 + 1 + 1 - \frac{3}{3} = 3 - 1 = 2.$

5. (56)

$f(\alpha,\beta) = \frac{\beta^3}{2}\text{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$



Let $\tan^{-1}\left(\frac{\beta}{\alpha}\right) = \theta$ and $\tan^{-1}\left(\frac{\alpha}{\beta}\right) = \phi$

$f(\alpha,\beta) = \frac{\beta^3}{2\sin^2\frac{\theta}{2}} + \frac{\alpha^3}{2\cos^2\frac{\phi}{2}} = \frac{\beta^3}{1-\cos\theta} + \frac{\alpha^3}{1+\cos\phi}$

$= \frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}} + \frac{\alpha^3}{1+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$

$$= \sqrt{\alpha^2+\beta^2}\left[\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha}+\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right]$$

$$= \sqrt{\alpha^2+\beta^2}\left[\frac{\beta^3\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)}{\beta^2}+\frac{\alpha^3\left(\sqrt{\alpha^2+\beta^2}-\beta\right)}{\alpha^2}\right]$$

$$= \sqrt{\alpha^2+\beta^2}\left[\beta\sqrt{\alpha^2+\beta^2}+\alpha\sqrt{\alpha^2+\beta^2}\right]$$

$f(\alpha, \beta) = (\alpha^2+\beta^2)(\alpha+\beta)$

Now, $\alpha+\beta = 4$ and $\alpha\beta = 1$

$f(\alpha, \beta) = ((\alpha+\beta)^2 - 2\alpha\beta))(\alpha+\beta)$

$= (16-2)(4) = 56$

6. **(0)** $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

$$= \cos^{-1}\left[\cos\left(2\pi-\frac{\pi}{3}\right)\right]+\sin^{-1}\left[\sin\left(2\pi-\frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{\pi}{3}\right]+\sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3}+\left(-\frac{\pi}{3}\right) = 0$$

7. **(0)** $\cos\left\{\cos^{-1}\left(\frac{-1}{7}\right)+\sin^{-1}\left(\frac{-1}{7}\right)\right\} = \cos\frac{\pi}{2} = 0$.

$$\left[\because \cos^{-1}x+\sin^{-1}x = \frac{\pi}{2}\right]$$

8. **(0.5)** Given: $4\sin^{-1}x+\cos^{-1}x = \pi$

We know $\sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}$

So, $3\sin^{-1}x+\sin^{-1}x+\cos^{-1}x = \pi$

$\Rightarrow\ 3\sin^{-1}x = \pi-\frac{\pi}{2} = \frac{\pi}{2}\ \Rightarrow\ \sin^{-1}x = \frac{\pi}{6}$

$\Rightarrow\ x = \sin\frac{\pi}{6} = \frac{1}{2}$

Chapter-19: Matrices

1. **(0)** $A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \Rightarrow |A| = 1$

$$\text{adj}(A) = \begin{bmatrix}+\cos\theta & -\sin\theta\\ +\sin\theta & +\cos\theta\end{bmatrix}^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} = B$$

$$B^2 = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$$

$$\Rightarrow B^3 = \begin{bmatrix}\cos 3\theta & \sin 3\theta\\ -\sin 3\theta & \cos 3\theta\end{bmatrix}$$

$$\Rightarrow A^{-50} = B^{50} = \begin{bmatrix}\cos(50\theta) & \sin(50\theta)\\ -\sin(50\theta) & \cos(50\theta)\end{bmatrix}$$

$$(A^{-50})_{\theta=\frac{\pi}{12}} = \begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$\left[\because \cos\left(\frac{50\pi}{12}\right) = \cos\left(4\pi+\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

Then, $a+b+c-d = \frac{\sqrt{3}}{2}+\frac{1}{2}-\frac{1}{2}-\frac{\sqrt{3}}{2} = 0$

2. **(5)** Given that $10B = \begin{bmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{bmatrix}$

$$\Rightarrow B = \frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{bmatrix}$$

Also since, $B = A^{-1} \Rightarrow AB = I$

$$\Rightarrow \frac{1}{10}\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \frac{1}{10}\begin{bmatrix}10 & 0 & 5-\alpha\\ 0 & 10 & -5+\alpha\\ 0 & 0 & 5+\alpha\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \frac{5-\alpha}{10} = 0 \Rightarrow \alpha = 5$$

3. **(198)** $\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix}$

Sum of diagonal elements,

$a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2 = 5$

Case – I: Five (1's) and four (0's)

$^9C_5 = 126$

Case – II: One (2) and one (1)

$^9C_2 = 2! = 72$

$\therefore$ Total = 198

4. (10) $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 6 & 1 \end{bmatrix}$

$\Rightarrow P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 90 & 12 & 1 \end{bmatrix}$

$\Rightarrow P^5 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 90 & 12 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 135 & 15 & 1 \end{bmatrix}$

$\because Q - P^5 = I_3$

$\therefore Q = I_3 + P^5 = \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix}$

$\dfrac{q_{21} + q_{31}}{q_{32}} = \dfrac{15 + 135}{15} = 10$

5. (5) $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$

$\because A^2$ is the unit matrix $\Rightarrow x^2 + 1 = 1 \Rightarrow x = 0$

$\therefore x^3 + x + 2 = 2.$

6. (20) Given, $f(x) = x^2 + 4x - 5I$ and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

$f(A) = A^2 + 4A - 5I$

Now, $A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

$4A = 4\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix}$

and $5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\Rightarrow f(A) = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix}$

$\therefore$ Sum of all elements of $f(A) = 8 + 4 + 8 + 0 = 20$

7. (6) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$

Similarly, $A^4 = A^2 . A^2 = \begin{bmatrix} \cos 4\alpha & -\sin 4\alpha \\ \sin 4\alpha & \cos 4\alpha \end{bmatrix}$

and so on $A^{32} = \begin{bmatrix} \cos 32\alpha & -\sin 32\alpha \\ \sin 32\alpha & \cos 32\alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Then $\sin 32\alpha = 1$ and $\cos 32\alpha = 0$

$32\alpha = n\pi + (-1)^n \dfrac{\pi}{2}$ and $32\alpha = 2n\pi + \dfrac{\pi}{2}$

$\alpha = \dfrac{n\pi}{32} + (-1)^n \dfrac{\pi}{64}$ and $\alpha = \dfrac{n\pi}{16} + \dfrac{\pi}{64}$ where $n \in Z$

Put $n = 0$, $\alpha = \dfrac{\pi}{64} = \dfrac{\pi}{2^6}$

Then, the value of α is $\dfrac{\pi}{2^6} = \dfrac{\pi}{2^k} \Rightarrow k = 6$

8. (1.25) $\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = O$

$\Rightarrow \begin{bmatrix} 1 & 2+5x+3 & 3+x+2 \end{bmatrix}\begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = O$

$\Rightarrow x + (2 + 5x + 3) + (-2)(3 + x + 2) = 0$

$\Rightarrow x = \dfrac{5}{4}.$

Chapter-20 : Determinants

1. (3) Since the system of linear equations are

$x + y + z = 2$ ...(1)

$2x + 3y + 2z = 5$ ...(2)

$2x + 3y + (a^2 - 1)z = a + 1$ ...(3)

Now, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & a^2-1 \end{vmatrix}$

$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 0 & a^2-3 \end{vmatrix}$

(Applying $R_3 \rightarrow R_3 - R_2$)

$= a^2 - 3$

When, $\Delta = 0 \Rightarrow a^2 - 3 = 0 \Rightarrow |a| = \sqrt{3}$

If $a^2 = 3$, then plane represented by eqn (2) and eqn (3) are parallel.

Hence, the given system of equation is inconsistent for $a^2 = 3$.

2. **(0.2)** Given $A=\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}$ and $|A^2|=25$

$$\therefore\ A^2=\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}\begin{bmatrix}5 & 5\alpha & \alpha\\ 0 & \alpha & 5\alpha\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}25 & 25\alpha+5\alpha^2 & 5\alpha+25\alpha^2+5\alpha\\ 0 & \alpha^2 & 5\alpha^2+25\alpha\\ 0 & 0 & 25\end{bmatrix}$$

$$\therefore\ |A^2|=25\ (25\alpha^2)$$

$$\therefore\ 25=25\ (25\alpha^2)\ \Rightarrow\ |\alpha|=\frac{1}{5}$$

3. **(2)** Since, the system of linear equations has, non-trivial solution then determinant of coefficient matrix = 0

$$\text{i.e.,}\begin{vmatrix}\sin 3\theta & \cos 2\theta & 2\\ 1 & 3 & 7\\ -1 & 4 & 7\end{vmatrix}=0$$

$\sin3\theta(21-28)-\cos2\theta(7+7)+2\ (4+3)=0$

$\sin3\theta+2\cos2\theta-2=0$

$3\sin\theta-4\sin^3\theta+2-4\sin^2\theta-2=0$

$4\sin^3\theta+4\sin^2\theta-3\sin\theta=0$

$\sin\theta\ (4\sin^2\theta+4\sin\theta-3)=0$

$\sin\theta\ (4\sin^2\theta+6\sin\theta-2\sin\theta-3)=0$

$\sin\theta\ [2\sin\theta\ (2\sin\theta-1)+3\ (2\sin\theta-1)]=0$

$\sin\theta\ (2\sin\theta-1)\ (2\sin\theta+3)=0$

$\sin\theta=0,\ \sin\theta=\frac{1}{2}$ $\qquad\left(\because \sin\theta\neq-\frac{3}{2}\right)$

$\theta=\frac{\pi}{6},\frac{5\pi}{6}$

Hence, for two values of θ, system of equations has non-trivial solution

4. **(0)** $\Delta=\begin{vmatrix}1 & \omega^n & \omega^{2n}\\ \omega^n & \omega^{2n} & 1\\ \omega^{2n} & 1 & \omega^n\end{vmatrix}$

$$=1\left(\omega^{3n}-1\right)-\omega^n\left(\omega^{2n}-\omega^{2n}\right)+\omega^{2n}\left(\omega^n-\omega^{4n}\right)$$

$$=\omega^{3n}-1-0+\omega^{3n}-\omega^{6n}$$

$$=1-1+1-1=0\left[\because \omega^{3n}=1\right]$$

5. **(2)** Applying, $C_1\to C_1+C_2+C_3$ we get

$$f(x)=\begin{vmatrix}1+(a^2+b^2+c^2+2)x & (1+b^2)x & (1+c^2)x\\ 1+(a^2+b^2+c^2+2)x & 1+b^2x & (1+c^2x)\\ 1+(a^2+b^2+c^2+2)x & (1+b^2)x & 1+c^2x\end{vmatrix}$$

$$=\begin{vmatrix}1 & (1+b^2)x & (1+c^2)x\\ 1 & 1+b^2x & (1+c^2x)\\ 1 & (1+b^2)x & 1+c^2x\end{vmatrix}$$

[As given that $a^2+b^2+c^2=-2$]

$\therefore\ a^2+b^2+c^2+2=0$

Applying $R_1\to R_1-R_2,\ R_2\to R_2-R_3$

$$\therefore\ f(x)=\begin{vmatrix}0 & x-1 & 0\\ 0 & 1-x & x-1\\ 1 & (1+b^2)x & 1+c^2x\end{vmatrix}$$

$f(x)=(x-1)^2$

Hence, degree of polynomial f(x) = 2.

6. **(1.5)** $|A|=\begin{vmatrix}2 & b & 1\\ b & b^2+1 & b\\ 1 & b & 2\end{vmatrix}$

$=2(2b^2+2-b^2)-b(2b-b)+1(b^2-b^2-1)$

$=2b^2+4-b^2-1=b^2+3$

$$\frac{|A|}{b}=b+\frac{3}{b}$$

$$\because\ \frac{b+\frac{3}{b}}{2}\geq\left(b\frac{3}{b}\right)^{\frac{1}{2}}\Rightarrow b+\frac{3}{b}\geq2\sqrt{3}$$

$$\therefore\ \frac{|A|}{b}\geq2\sqrt{3}$$

Minimum value of $\frac{|A|}{b}$ is $2\sqrt{3}=2\times1.73=3.46$

7. **(0.5)** If the system of equations has non-trivial solutions, then the determinant of coefficient matrix is zero

$$\begin{vmatrix}1 & -c & -c\\ c & -1 & c\\ c & c & -1\end{vmatrix}=0$$

$(1-c^2)+c\ (-c-c^2)-c\ (c^2+c)=0$

$(1+c)\ (1-c)-2c^2\ (1+c)=0$

$(1+c)\ (1-c-2c^2)=0$

$(1+c)^2\ (1-2c)=0$

$c=-1$ or $\frac{1}{2}$

Hence, the greatest value of c is $\frac{1}{2}$ for which the system of linear equations has non-trivial solution.

8. **(0)** $\because\ a_1, a_2, a_3,.......$ are in G.P.

$\therefore$ Using $a_n=ar^{n-1}$, we get the given determinant, as

$$\begin{vmatrix} \log ar^{n-1} & \log ar^{n} & \log ar^{n+1} \\ \log ar^{n+2} & \log ar^{n+3} & \log ar^{n+4} \\ \log ar^{n+5} & \log ar^{n+6} & \log ar^{n+7} \end{vmatrix}$$

Operating $C_3 - C_2$ and $C_2 - C_1$ and using

$\log m - \log n = \log \frac{m}{n}$ we get

$$= \begin{vmatrix} \log ar^{n-1} & \log r & \log r \\ \log ar^{n+2} & \log r & \log r \\ \log ar^{n+5} & \log r & \log r \end{vmatrix}$$

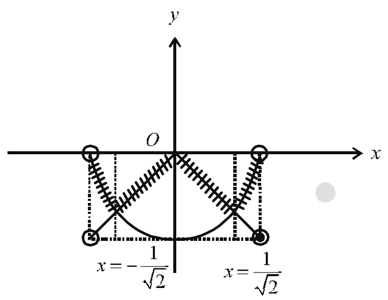
= 0 (two columns being identical)

Chapter-21 : Continuity and Differentiability

1. **(3)** Consider the function

$$f(x) = \max\left\{-|x|, -\sqrt{1-x^2}\right\}$$

Now, the graph of the function



From the graph, it is clear that $f(x)$ is not differentiable at $x = 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Then, $K = \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

Hence, K has exactly three elements.

2. **(5)** $f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$;

As function is differentiable so it is continuous as it is given that $\lim_{h \to 0} \frac{f(1+h)}{h} = 5$ and hence $f(1) = 0$

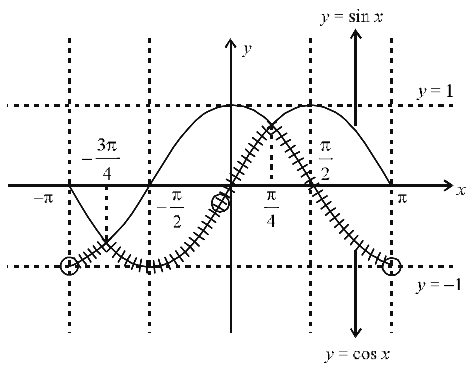
Hence, $f'(1) = \lim_{h \to 0} \frac{f(1+h)}{h} = 5$

3. **(4)** $\lim_{x \to a} \frac{f(a)g'(x) - g(a)f'(x)}{g'(x) - f'(x)} = 4$

(By L' Hospital rule)

$\lim_{x \to a} \frac{k\,g'(x) - k\,f'(x)}{g'(x) - f'(x)} = 4 \qquad \therefore k = 4.$

4. **(2)** $f(x) = \min\{\sin x, \cos x\}$



$\because$ $f(x)$ is not differentiable at $x = -\frac{3\pi}{4}, \frac{\pi}{4}$

$\therefore$ $S = \left\{-\frac{3\pi}{4}, \frac{\pi}{4}\right\}$

5. **(6)** $f(\pi/2) = 3$. Since $f(x)$ is continuous at $x = \pi/2$

$\Rightarrow \lim_{x \to \pi/2} \left(\frac{k \cos x}{\pi - 2x}\right) = f\left(\frac{\pi}{2}\right)$

$\Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$

6. **(1)** $\lim_{x \to 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x \to 0} \frac{1}{x} \log(\cos x) = \log k$

$\Rightarrow \lim_{x \to 0} \frac{1}{x} \lim_{x \to 0} \log \cos x = \log k$

$\Rightarrow \lim_{x \to 0} \frac{1}{x} \times 0 = \log_e k \Rightarrow k = 1.$

7. **(0.5)** Since, f(x) is continuous, then

$\lim_{x \to \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$

$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1} = k$

Now by L- Hospital's rule

$\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\sin x}{\operatorname{cosec}^2 x} = k \Rightarrow \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}{(\sqrt{2})^2} = k \Rightarrow k = \frac{1}{2}$

8. **(2)** $(x^2 - 3x + 2) = (x - 1)(x - 2) = +$ve, when $x < 1$ or $x > 2 = -$ve, when $1 \le x \le 2$

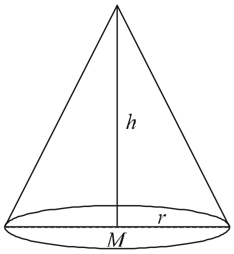
Also $\cos|x| = \cos x$ $\quad [\because \cos(-x) = \cos x]$

$\therefore f(x) = \begin{cases} -(x^2-1)(x^2-3x+2) + \cos x, & 1 \le x \le 2 \\ (x^2-1)(x^2-3x+2) + \cos x, & x > 2 \end{cases}$

Evidently f(x) is not differentiable at x = 2 as L.H.D. ≠ R.H.D.

Chapter-22: Application of Derivatives

1. **(2)**



$h^2 + r^2 = 3^2 = 9$ ...(1)

Volume of cone

$V = \frac{1}{3}\pi r^2 h$ ...(2)

From (1) and (2),

$\Rightarrow V = \frac{1}{3}\pi(9-h^2)h$

$\Rightarrow V = \frac{1}{3}\pi(9h-h^3) \Rightarrow \frac{dv}{dh} = \frac{1}{3}\pi(9-3h^2)$

For maxima/minima,

$\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi(9-3h^2) = 0$

$\Rightarrow h = \pm\sqrt{3} \Rightarrow h = \sqrt{3}$ $(\because h > 0)$

Now; $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(-6h) = -2\pi h = -2\sqrt{3}\pi$

Here, $\left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\sqrt{3}} < 0$

Then, $h = \sqrt{3}$ is point of maxima

Hence, the required maximum volume is,

$V = \frac{1}{3}\pi(9-3)\sqrt{3} = 2\sqrt{3}\pi \text{ m}^3$

$\Rightarrow k\sqrt{3}\pi = 2\sqrt{3}\pi \Rightarrow k = 2$

2. **(1)** $\because$ Slope of the tangent $= \frac{x^2-2y}{x}$

$\because \frac{dy}{dx} = \frac{x^2-2y}{x}$

$\frac{dy}{dx} + \frac{2}{x}y = x$

I.F. $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Solution of equation

$y \cdot x^2 = \int x \cdot x^2 dx$

$x^2 y = \frac{x^4}{4} + C$

$\therefore$ curve passes through point (1, –2)

$(1)^2(-2) = \frac{1^4}{4} + C$

$\Rightarrow C = \frac{-9}{4}$

Then, equation of curve

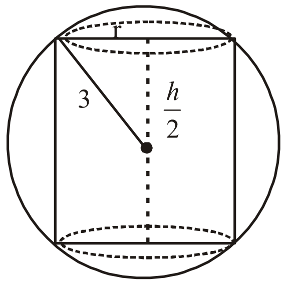
$y = \frac{x^2}{4} - \frac{9}{4x^2}$

Since, above curve satisfies the point.

Hence, the curve passes through $(\sqrt{3}, 0)$.

$\Rightarrow \sqrt{3}a = \sqrt{3} \Rightarrow a = 1$

3. **(2)** Let radius of base and height of cylinder be r and h respectively.



$\therefore r^2 + \frac{h^2}{4} = 9$ ...(i)

Now, volume of cylinder, $V = \pi r^2 h$

Substitute the value of r^2 from equation (i),

$V = \pi h\left(9 - \frac{h^2}{4}\right) \Rightarrow V = 9\pi h - \frac{\pi}{4}h^3$

Differentiating w.r.t. h,

$\frac{dV}{dh} = 9\pi - \frac{3}{4}\pi h^2$

For maxima/minima,

$\frac{dV}{dh} = 0 \Rightarrow h = \sqrt{12}$

and $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$

$\therefore \left(\frac{d^2V}{dh^2}\right)_{h=\sqrt{12}} < 0$

$\Rightarrow$ Volume is maximum when $h = 2\sqrt{3}$

$\Rightarrow m\sqrt{3} = 2\sqrt{3} \Rightarrow m = 2$

4. **(2)** $\frac{x}{2} + \frac{2}{x}$ is of the form $y + \frac{1}{y}$ where $y + \frac{1}{y} \geq 2$ and equality holds for $y = 1$

$\therefore$ Min value of function occurs at $\frac{x}{2} = 1$ i.e., at $x = 2$

5. (3.5) Given curve $y^2 = px^3 + q$ ...(i)

Differentiate with respect to x,

$$2y.\frac{dy}{dx} = 3px^2 \Rightarrow \frac{dy}{dx} = \frac{3p}{2}\left(\frac{x^2}{y}\right)$$

$$\therefore \left|\frac{dy}{dx}\right|_{(2,3)} = \frac{3p}{2} \times \frac{4}{3} = 2p$$

For given line, slope of tangent = 4

$\therefore 2p = 4 \Rightarrow p = 2$

From equation (i), $9 = 2\times 8 + q \Rightarrow q = -7$

Hence, $\left|\frac{q}{p}\right| = \frac{7}{2}$

6. (4) $\because y^3 = 8x \Rightarrow 3y^2\frac{dy}{dx} = 8 \Rightarrow \frac{dy}{dx} = \frac{8}{3y^2}$ ...(i)

Also, $y^2 = 12 - \frac{12x^2}{a^2} \Rightarrow 2y\frac{dy}{dx} = \frac{-24x}{a^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{-12x}{a^2y} \quad \text{...(ii)}$$

$$\therefore \left(\frac{-12x}{a^2y}\right)\left(\frac{8}{3y^2}\right) = -1 \quad [\because \text{Intersect at right angles}]$$

$$\Rightarrow a^2y^3 = 32x \Rightarrow a^2 = 4 \quad [\because y^3 = 8x]$$

7. (12250) $s = 490t - 4.9t^2 \Rightarrow \frac{ds}{dt} = -9.8t + 490$

$\Rightarrow 0 = 490 - 9.8t \Rightarrow t = 50$

$\therefore s = 490 \times 50 - 4.9(50)^2 = 12250$

8. (12) Let $f(x) = 4e^{2x} + 9e^{-2x}$

$\therefore f'(x) = 8e^{2x} - 18e^{-2x}$

Put $f'(x) = 0 \Rightarrow 8e^{2x} - 18e^{-2x} = 0$

$e^{2x} = 3/2 \Rightarrow x = \log(3/2)^{1/2}$

Again $f''(x) = 16e^{2x} + 36e^{-2x} > 0$

Now $f\left(\log(3/2)^{1/2}\right) = 4e^{2\left(\log(3/2)^{1/2}\right)} + 9e^{-2\left(\log(3/2)^{1/2}\right)}$

$= 4\times\frac{3}{2} + 9\times\frac{2}{3} = 6 + 6 = 12$

Hence, minimum value = 12.

Chapter-23 : Integrals

1. (1) $\because f : R \to R$

and $|f(x) - f(y)| \le 2 \cdot |x - y|^{3/2}$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \le 2\sqrt{x-y}$$

$$\Rightarrow \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{x\to y} 2\sqrt{x-y}$$

$\Rightarrow |f'(x)| = 0$

$\therefore f(x)$ is a constant function.

Given $f(0) = 1 \Rightarrow f(x) = 1$

Hence, the integral

$$\int_0^1 f^2(x)\,dx = \int_0^1 1\,dx = [x]_0^1 = 1$$

2. (4) $I = \int_0^{\pi} |\cos x|^3\,dx = 2\int_0^{\pi/2} \cos^3 x\,dx$

$$= \frac{2}{4}\int_0^{\pi/2}(3\cos x + \cos 3x)dx$$

$[\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$

$$= \frac{1}{2}\left[3\sin x + \frac{\sin 3x}{3}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(3 - \frac{1}{3}\right) = \frac{4}{3}$$

$\Rightarrow \frac{k}{3} = \frac{4}{3}$

$\Rightarrow k = 4$

3. (2) $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$

$\Rightarrow f(x) = 2x - x^2 f(x)$

$\Rightarrow f(x) = \frac{2x}{1+x^2}$

$\Rightarrow f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}$

Then,

$$f'(1/2) = \frac{2\left(1-\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)^2} = \frac{3}{2}\times\frac{16}{25} = \frac{24}{25}$$

$\Rightarrow \frac{24}{5^k} = \frac{24}{25}$

$\Rightarrow \frac{24}{5^k} = \frac{24}{5^2}$

$\Rightarrow k = 2$

4. (1) $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x(1 + \tan^2 x)dx$

$$= \int_0^{\pi/4} \tan^n x \sec^2 x\,dx = \left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\pi/4}$$

$$= \frac{1-0}{n+1} = \frac{1}{n+1}$$

$\therefore\ I_n + I_{n+2} = \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} n\,[I_n + I_{n+2}]$

$= \lim_{n\to\infty} n.\frac{1}{n+1} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} = 1$

5. **(0.2)** $\lim_{n\to\infty} \frac{1^4+2^4+3^4+.......n^4}{n^5}$

$- \lim_{n\to\infty} \frac{1^3+2^3+3^3+.......+n^3}{n^5}$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4 - \lim_{n\to\infty}\frac{1}{n}.\lim_{n\to\infty}\frac{1}{n}\left(\frac{r}{n}\right)^3$

$= \int_0^1 x^4\,dx - \lim_{n\to\infty}\frac{1}{n}\times\int_0^1 x^3\,dx = \left[\frac{x^5}{5}\right]_0^1 - 0 = \frac{1}{5} = 0.2$

6. **(0.5)** Given f is a function and

$I_1 = \int_{1-k}^{k} x\, f\left[x(1-x)\right] dx$

$I_2 = \int_{1-k}^{k} f\left[x(1-x)\right] dx$

Now, $I_1 = \int_{1-k}^{k} x\, f\left[x(1-x)\right] dx$ ...(i)

$\int_{1-k}^{k} (1-x)\, f\left[(1-x)x\right] dx$ ...(ii)

$\left[\text{Using Property } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

Adding (i) and (ii)

$2I_1 = \int_{1-k}^{k} f\left[x(1-x)\right] dx = I_2$

$\therefore \frac{I_1}{I_2} = \frac{1}{2} = 0.5$

7. **(0)** We have, $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[f(x)+f(-x)\right]\ \left[g(x)-g(-x)\right] dx$

Let $F(x) = \left(f(x)+f(-x)\right)\ \left(g(x)-g(-x)\right)$, then

$F(-x) = \left(f(-x)+f(x)\right)\left(g(-x)-g(x)\right)$

$= -\left[f(x)+f(-x)\right]\left[g(x)-g(-x)\right]$

$= -F(x)$

$\therefore$ $F(x)$ is an odd function.

Using the property $\int_{-a}^{a} f(x)\,dx = 0$,

if $f(-x) = f(x)$, we get $I = 0$

8. **(1)** Let, $\int \frac{dx}{x^3(1+x^6)^{\frac{2}{3}}} = \int \frac{dx}{x^7(1+x^{-6})^{\frac{2}{3}}}$

Put $1 + x^{-6} = t^3 \Rightarrow -6^{-7}\,dx = 3t^2\,dt$

$\Rightarrow \frac{dx}{x^7} = \left(-\frac{1}{2}\right)t^2 dt$

Now, $I = \int\left(-\frac{1}{2}\right)\frac{t^2 dt}{t^2} = -\frac{1}{2}t + C$

$= -\frac{1}{2}(1+x^{-6})^{\frac{1}{3}} + C = -\frac{1}{2}\frac{(1+x^6)^{\frac{1}{3}}}{x^2} + C$

$= -\frac{1}{2x^3}x(1+x^6)^{\frac{1}{3}} + C$

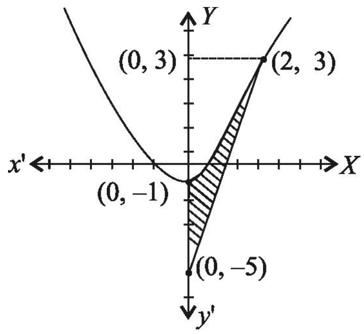
Hence, $f(x) = -\frac{1}{2x^3}$

$\Rightarrow \frac{-C}{2x^3} = \frac{-1}{2x^3}$

$\Rightarrow C = 1$

Chapter-24 : Application of Integrals

1. **(2.6)**



Curve is given as : $y = x^2 - 1$

$\Rightarrow \frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = 4$

$\therefore$ equation of tangent at (2, 3)

$(y-3) = 4(x-2) \Rightarrow y = 4x - 5$

but $x = 0 \Rightarrow y = -5$

Here the curve cuts Y–axis

$\therefore$ required area

$= \frac{1}{4}\int_{-5}^{3}(y+5)dy - \int_{-1}^{3}\sqrt{y+1}\,dy$

$= \frac{1}{4}\left[\frac{y^2}{2}+5y\right]_{-5}^{3} \frac{-2}{3}\left[(y+1)^{3/2}\right]_{-1}^{3}$

$= \frac{1}{4}\left[\frac{9}{2}+15-\frac{25}{2}+25\right] - \frac{2}{3}\left[4^{3/2}-0\right]$

$= \frac{32}{4} - \frac{16}{3} = \frac{8}{3}$ sq-units.

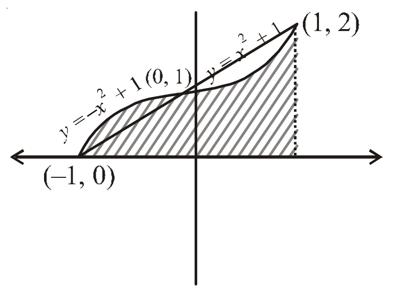
2. **(0.5)** We have, $\int_0^a f(x)\,dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

Differentiating w.r.t. a, we get

$f(a) = a + \frac{1}{2}(\sin a + a\cos a) - \frac{\pi}{2}\sin a$

Put $a = \frac{\pi}{2}$; $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$

3. **(2)** Given $A = \{(x, y) : 0 \le y \le x|x| + 1$ and $-1 \le x \le 1\}$



$\therefore$ Area of shaded region

$= \int_{-1}^{0}(-x^2+1)dx + \int_{0}^{1}(x^2+1)dx$

$= \left(-\frac{x^3}{3}+x\right)_{-1}^{0} + \left(\frac{x^3}{3}+x\right)_{0}^{1}$

$= 0 - \left(\frac{1}{3}-1\right) + \left(\frac{1}{3}+1\right) - (0+0)$

$= \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$ square units

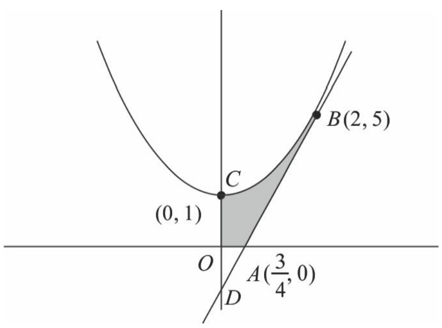
4. **(1)** Required area $= \int_{1}^{3}|x-2|dx$

$= \int_{1}^{2}(2-x)\,dx + \int_{2}^{3}(x-2)\,dx$

$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3$

$= \left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right] + \left[\frac{9}{2} - 6 - \frac{4}{2} + 4\right] = \frac{1}{2} + \frac{1}{2} = 1.$

5. **(1.54)**



The equation of parabo la $x^2 = y - 1$

The equation of tangent at (2, 5) to parabola is

$y - 5 = \left(\frac{dy}{dx}\right)_{(2,5)}(x - 2)$

$y - 5 = 4(x - 2)$

$4x - y = 3$

Then, the required area

$= \int_0^2 \{(x^2+1) - (4x-3)\}\,dx$ – Area of ΔAOD

$= \int_0^2 (x^2 - 4x + 4)\,dx - \frac{1}{2} \times \frac{3}{4} \times 3$

$= \left[\frac{(x-2)^3}{3}\right]_0^2 - \frac{9}{8} = \frac{37}{24}$

6. **(2)** Given $|x| + |y| = 1 \Rightarrow \pm x \pm y = 1$

Thus the equation of its sides are

$x + y = 1$ ...(i)

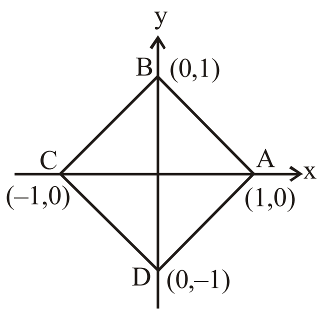
$x - y = 1$ ...(ii)

$-x + y = 1$ ...(iii)

$x + y = -1$ ...(iv)

From the given equation, we have,

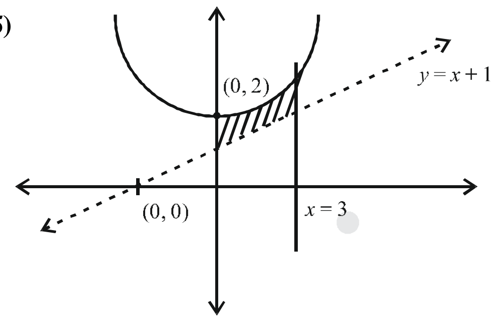
Four vertices of square, (0, 1), (1, 0), (0, –1) and (–1, 0)



One side of square $= \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$.

$\therefore$ Area of square $= (\text{side})^2 = (\sqrt{2})^2 = 2$ sq. unit

7. **(7.5)**



Area of the bounded region $\int_0^3[(x^2+2) - (x+1)]dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^3 = 9 - \frac{9}{2} + 3 = \frac{15}{2}$

8. **(2)** Given curves $y^2 = 4ax$ and $y = mx$ intersect at $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$. The area enclosed by the two curves is

$$= \int_0^{4a/m^2} (\sqrt{4ax} - mx)\,dx.$$

Given $\int_0^{4a/m^2} (\sqrt{4ax} - mx)\,dx = \frac{a^2}{3}$

$$\Rightarrow \left[\frac{4\sqrt{a}}{3}\cdot x^{\frac{3}{2}} - \frac{m}{2}x^2\right]_0^{\frac{4a}{m^2}} = \frac{a^2}{3}$$

$$\Rightarrow \frac{8}{3}\frac{a^2}{m^3} = \frac{a^2}{3} \Rightarrow m^3 = 8 \Rightarrow m = 2.$$

Chapter-25: Differential Equations

1. (12.25) Since, $x\frac{dy}{dx} + 2y = x^2$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x$$

I.F. $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$

Solution of differential equation is:

$$y \,.\, x^2 = \int x \cdot x^2 dx$$

$$y \,.\, x^2 = \frac{x^4}{4} + C \qquad ...(1)$$

$\because \; y(1) = 1$

$\therefore \; C = \frac{3}{4}$

Then, from equation (1)

$$y \,.\, x^2 = \frac{x^4}{4} + \frac{3}{4}$$

$$\therefore \; y = \frac{x^2}{4} + \frac{3}{4x^2} \qquad \therefore \; y\left(\frac{1}{2}\right) = \frac{1}{16} + 3 = \frac{49}{16}$$

$$\Rightarrow 4y\left(\frac{1}{2}\right) = \frac{4\times 49}{16} = 12.25$$

2. (3) $f(xy) = f(x).f(y)$ ...(1)

Put $x = y = 0$ in (1) to get $f(0) = 1$

Put $x = y = 1$ in (1) to get $f(1) = 0$ or $f(1) = 1$

$f(1) = 0$ is rejected else $y = 1$ in (1) gives $f(x) = 0$ imply $f(0) = 0$.

Hence, $f(0) = 1$ and $f(1) = 1$

By first principle derivative formula,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} f(x)\left(\frac{f\left(1+\frac{h}{x}\right) - f(1)}{h}\right)$$

$$\Rightarrow f'(x) = \frac{f(x)}{x}f'(1)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{k}{x} \Rightarrow \ln f(x) = k\ln x + c$$

$f(1) = 1 \Rightarrow \ln 1 = k\ln 1 + c \Rightarrow c = 0$

$\Rightarrow \ln f(x) = k\ln x \Rightarrow f(x) = x^k$ but $f(0) = 1$

$\Rightarrow k = 0$

$\therefore \; f(x) = 1$

$\frac{dy}{dx} = f(x) = 1 \Rightarrow y = x + c,\; y(0) = 1 \Rightarrow c = 1$

$\Rightarrow y = x + 1$

$$\therefore \; y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right) = \frac{1}{4} + 1 + \frac{3}{4} + 1 = 3$$

3. (4) Consider the differential equation,

$$\frac{dy}{dx} + \frac{y}{x} = \log_e x$$

$\because$ I.F. $= e^{\int \frac{1}{x}dx} = x$

$\therefore \; yx = \int x\ln x\,dx$

$$\Rightarrow xy = \ln x\cdot\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$\Rightarrow xy = \frac{x^2}{2}\cdot\ln x - \frac{x^2}{4} + c$$

Given, $2y(2) = \log_e 4 - 1.$

$\therefore \; 2y = 2\ln 2 - 1 + c$

$\Rightarrow \ln 4 - 1 = \ln 4 - 1 + c$

i.e. $c = 0$

$$\Rightarrow xy = \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

$$\Rightarrow y = \frac{x}{2}\ln x - \frac{x}{4}$$

$$\Rightarrow y(e) = \frac{e}{2} - \frac{e}{4} = \frac{e}{4}$$

$$\Rightarrow \frac{e}{4} = \frac{e}{k} \Rightarrow k = 4$$

4. (1) $y = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... + \infty$ where $t = \frac{dy}{dx}$

$\Rightarrow y = e^t,$

$\therefore t = \log y \Rightarrow \frac{dy}{dx} = \log y.$

Hence degree is 1.

5. (5) Given, $\left(1+y_1^2\right)^{2/3} = y_2 \Rightarrow \left(1+y_1^2\right)^2 = y_2^3$

So, order $n = 2$ and degree $m = 3$

Thus, $\frac{m+n}{m-n} = \frac{5}{1} = 5.$

6. **(8)** Let $\frac{d^2y}{dx^2}=t\Rightarrow\frac{d^3y}{dx^3}=\frac{dt}{dx}$ and the given equation recuces to $\frac{dt}{dx}=8t$.

Separating the variables, $\frac{dt}{t}=8dx$. Integrating we get.

$\ell n\, t=8x+c_1\Rightarrow \ell n\, y''=8x+C_1$

Put $x=0$, then $y''=1\Rightarrow C_1=0$

$\therefore \ell n\, y''=8x\Rightarrow y''=e^{8x}$,

Again integrate, we get $y'=\frac{e^{8x}}{8}+C_2$

Again putting $x=0$ and $y'=0\Rightarrow C_2=-\frac{1}{8}$

$\therefore y'=\frac{e^{8x}}{8}-\frac{1}{8}\Rightarrow y=\frac{1}{8}\left(\frac{e^{8x}}{8}-x\right)+C_3$

[After integration]

By giving values $x=0$, and $y=\frac{1}{8}$, we get $C_3=\frac{7}{64}$

Hence the final solution is $y=\frac{1}{8}\left[\frac{e^{8x}}{8}-x+\frac{7}{8}\right]$

$\Rightarrow p=8$

7. **(0.5)** Given $\frac{dx}{dt}=\cos^2\pi x$. Differentiate *w.r.t. t*,

$\frac{d^2x}{dt^2}=-\pi\sin 2\pi x=-\text{ve}$

$\because \frac{d^2x}{dt^2}=0\Rightarrow -\pi\sin 2\pi x=0$

$\Rightarrow \sin 2\pi x=\sin\pi$

$\Rightarrow 2\pi x=\pi\Rightarrow x=1/2.\Rightarrow x=0.5$

8. **(2)** By multiplying e^{-t} and rearranging the terms, we get

$e^{-t}(1+t)dy+y(e^{-t}-(1+t)e^{-t})dt=e^{-t}dt$

$\Rightarrow d(e^{-t}(1+t)y)=d(-e^{-t})\Rightarrow ye^{-t}(1+t)=-e^{-t}+c.$

Also $y_0=-1\Rightarrow c=0\Rightarrow y(1)=-1/2$

$\Rightarrow \frac{-1}{k}=\frac{-1}{2}$

$\Rightarrow k=2$

Chapter-26 : Vector Algebra

1. **(9.5)** $|\vec a\times\vec c|^2=|\vec a|^2|\vec c|^2-(\vec a.\vec c)^2$

$\Rightarrow |-\vec b|^2=2|\vec c|^2-16\Rightarrow 3=2|\vec c|^2-16$

$\Rightarrow |\vec c|^2=\frac{19}{2}$

2. **(44)** Resultant $\vec F=7\hat i+4\hat j-4\hat k$.

$\vec d=(5\hat i+4\hat j+\hat k)-(\hat i+2\hat j+3\hat k)=4\hat i+2\hat j-2\hat k$

$\therefore W=\vec F.\vec d=(7\hat i+4\hat j-4\hat k)(4\hat i+2\hat j-2\hat k)$

$=28+8+8=44.$

3. **(6)** Projection of $\vec b$ on $\vec a$ = $\frac{\vec b\cdot\vec a}{|\vec a|}=\frac{b_1+b_2+2}{4}$

According to question $\frac{b_1+b_2+2}{2}=\sqrt{1+1+2}=2$

$\Rightarrow b_1+b_2=2$ ...(1)

Since, $\vec a+\vec b$ is perpendicular to $\vec c$.

Hence, $\vec a\cdot\vec c+\vec b\cdot\vec c=0$

$\Rightarrow 8+5b_1+b_2+2=0$ ...(2)

From (1) and (2),

$b_1=-3,\ b_2=5$

$\Rightarrow \vec b=-3\cdot\hat i+5\hat j+\sqrt2\hat k$

$|\vec b|=\sqrt{9+25+2}=6$

4. **(3)** since $\vec n$ is perpendicular $\vec u$ *and* $\vec v$, $\vec n=\frac{\vec u\times\vec v}{|\vec u||\vec v|}$

$$\hat n=\frac{\begin{vmatrix} i & j & k\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}}{\sqrt2\times\sqrt2}=\frac{-2\hat k}{2}=-\hat k$$

$|\vec w.\hat n|=|(i+2j+3k).(-k)|=|-3|=3$

5. **(4)** Let $\vec\alpha$ and $\vec\beta$ are collinear for same k

i.e., $\vec\alpha=k\vec\beta$

$(\lambda-2)\ \vec a+\vec b=k((4\lambda-2)\ \vec\alpha+3\vec\beta$

$(\lambda-2)\vec a+\vec b=k(4\lambda-2)\vec\alpha+3k\vec b$

$(\lambda-2-k(4\lambda-2))\vec\alpha+\vec\beta(1-3k)=0$

But $\vec\alpha$ and $\vec\beta$ are non-collinear, then

$\lambda-2-k(4\lambda-2)=0,\ 1-3k=0$

$\Rightarrow k=\frac{1}{3}$ and $\lambda-2-\frac{1}{3}(4\lambda-2)=0$

$3\lambda-6-4\lambda+2=0$

$\lambda=-4\Rightarrow|\lambda|=4$

6. **(1)** $|\vec x-\vec y|^2=(\vec x-\vec y).(\vec x-\vec y)$

Given, $|\vec x|=1, |\vec y|=1$

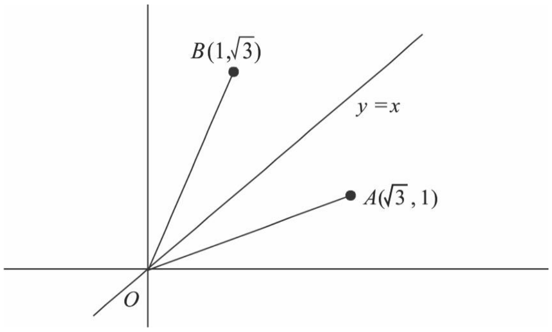
$\therefore|\vec x-\vec y|^2=|\vec x|^2+|\vec y|^2-2|\vec x||\vec y|\cos\pi$

$= 1 + 1 - 2|x||y|\cos\pi = 2 - 2\cos\pi$

$\therefore |\vec{x} - \vec{y}|^2 = 4$

So, $\frac{1}{2}|\vec{x} - \vec{y}| = 1$

7. **(1)** Since, the angle bisector of acute angle between *OA* and *OB* would be $y = x$



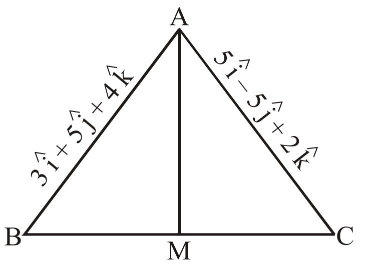
Since, the distance of *C* from bisector = $\frac{3}{\sqrt{2}}$

$\Rightarrow \left|\frac{\beta - (1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$ = $2\beta = \pm 3 + 1$

$\beta = 2$ or $\beta = -1$

Hence, the sum of all possible value of $\beta = 2 + (-1) = 1$

8. **(5)** Let the given vectors be $\overrightarrow{AB} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 5\hat{j} + 2\hat{k}$



Let AM be the median through A

$\therefore \overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

$= \frac{1}{2}[(3\hat{i} + 5\hat{j} + 4\hat{k}) + (5\hat{i} - 5\hat{j} + 2\hat{k})]$

$= \frac{1}{2}(8\hat{i} + 6\hat{k}) = (4\hat{i} + 3\hat{k})$

$\therefore$ Length of the median AM

$= \sqrt{4^2 + 3^2} = \sqrt{25} = 5$ units

Chapter-27 : Three Dimensional Geometry

1. **(0)** Let the direction ratios of the plane containing lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and is $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ $<a, b, c>$

$\therefore \quad 3a + 4b + 2c = 0$

$4a + 2b + 3c = 0$

$\therefore \quad \frac{a}{12-4} = \frac{b}{8-9} = \frac{c}{6-16}$

$\frac{a}{8} = \frac{b}{-1} = \frac{c}{-10}$

$\therefore$ Direction ratio of plane = $<-8, 1, 10>$.

Let the direction ratio of required plane is $<l, m, n>$

Then $-8l + m + 10n = 0$ ...(1)

and $2l + 3m + 4n = 0$ ...(2)

From (1) and (2),

$\frac{l}{-26} = \frac{m}{52} = \frac{n}{-26}$

$\therefore$ D.R.s are $<1, -2, 1>$

$\therefore$ Equation of plane: $x - 2y + z = 0$

$\Rightarrow px + qy + r = x - 2y + z$

$\Rightarrow p = 1, q = -2, r = 1$

Hence, $p + q + r = 1 - 2 + 1 = 0$

2. **(2.6)** If a line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube.
i.e. the equation becomes

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma + 1 - \sin^2\delta = \frac{4}{3}$

$(\because \cos^2\theta + \sin^2\theta = 1)$

$\Rightarrow -(\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta) = \frac{4}{3} - 4$

$\Rightarrow -(\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta) = -\frac{8}{3}$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = \frac{8}{3}$

3. **(4)** Since, $\alpha = \beta = \gamma \Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \alpha = \cos^{-1}\left(\pm\frac{1}{\sqrt{3}}\right)$

So, there are four lines whose direction cosines are

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),$$

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right).$$

4. **(7)** Let the normal to the required plane is , $\vec{n}$ then

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -4 & 4 \\ 2 & -5 & 0 \end{vmatrix} = 20\,\hat{i} + 8\,\hat{j} - 12\,\hat{j}$$

$\therefore$ Equation of the plane

$(x - 3) \times 20 + (y - 4) \times 8 + (z - 2) \times (-12) = 0$

$5x - 15 + 2y - 8 - 3z + 6 = 0$

$5x + 2y - 3z - 17 = 0$ ....(1)

Since, equation of plane (1) passes through $(2, \alpha, \beta)$, then $10 + 2\alpha - 3\beta - 17 = 0 \Rightarrow 2\alpha - 3\beta = 7$

5. **(4.5)** Any point on $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = \lambda$ is,

$(2\lambda + 1, 3\lambda - 1, 4\lambda + 1); \lambda \in R$

Any point on $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = \mu$ is,

$(\mu + 3, 2\mu + k, \mu); \mu \in R$

The given lines intersect if and only if the system of equations (in λ and μ)

$2\lambda + 1 = \mu + 3$ ...(i)

$3\lambda - 1 = 2\mu + k$ ...(ii)

$4\lambda + 1 = \mu$ ...(iii)

has a unique solution.

Solving (i) and (iii), we get $\lambda = \frac{-3}{2}, \mu = -5$

From (ii), we get

$$\frac{-9}{2} - 1 = -10 + k \Rightarrow k = \frac{9}{2}$$

6. **(66)** plane containing both lines.

$$\therefore \quad \text{D.R. of plane} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 7\hat{i} - 14\hat{j} + 7\hat{k}$$

Now, equation of plane is,

$7(x - 1) - 14(y - 4) + 7(z + 4) = 0$

$\Rightarrow\Rightarrow x - 1 - 2y + 8 + z + 4 = 0$

$\Rightarrow\Rightarrow x - 2y + z + 11 = 0$

Hence, distance from (0, 0, 0) to the plane,

$$= \frac{11}{\sqrt{1+4+1}} = \frac{11}{\sqrt{6}}$$

$\because$ The given distance $= \frac{p}{\sqrt{q}}$

$$\therefore \frac{p}{\sqrt{q}} = \frac{11}{\sqrt{6}} \Rightarrow p = 11, q = 6$$

Hence, $pq = 11 \times 6 = 66$

7. **(2)** The given lines are

$$x - 1 = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda} = s \quad \text{......(1)}$$

and $$2x = y - 1 = \frac{z-2}{-1} = t \quad \text{......(2)}$$

The lines are coplanar, if

$$\begin{vmatrix} 0-(-1) & -1-3 & -2-(-1) \\ 1 & -\lambda & \lambda \\ \frac{1}{2} & 1 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & -5 & -1 \\ 1 & 0 & \lambda \\ \frac{1}{2} & 0 & -1 \end{vmatrix} = 0 \quad (\text{applying } C_2 \to C_2 + C_3)$$

$$\Rightarrow 5(-1 - \frac{\lambda}{2}) = 0 \Rightarrow \lambda = -2 \Rightarrow |\lambda| = 2$$

8. **(4.5)** Let point on line be P (2k + 1, 3k – 1, 4k + 2)

Since, point P lies on the plane x + 2y + 3z = 15

$\therefore$ 2k + 1 + 6k – 2 + 12k + 6 = 15

$$\Rightarrow k = \frac{1}{2}$$

$$\therefore P \equiv \left(2, \frac{1}{2}, 4\right)$$

Then the distance of the point P from the origin is

$$OP = \sqrt{4 + \frac{1}{4} + 16} = \frac{9}{2}$$

Chapter-28: Probability – II

1. **(6.76)** X = number of aces drawn

$\therefore \quad P(X = 1) + P(X = 2)$

$$= \left\{\frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52}\right\} + \left\{\frac{4}{52} \times \frac{4}{52}\right\}$$

$$= \frac{24}{169} + \frac{1}{169} = \frac{25}{169} = k$$

$$\Rightarrow \frac{1}{k} = \frac{169}{25} = 6.76$$

2. (0.4) $P(E)=\frac{3}{4}$, $P(T)=1-\frac{3}{4}=\frac{1}{4}$

$F \rightarrow$ Fair coin, $T \rightarrow$ Two headed, $H \rightarrow$ Head occurs

$$\therefore P(T/H)=\frac{P(H/T).P(T)}{P(H/T).P(T)+P(H/F).P(F)}$$

[Using Baye's theorem]

$$=\frac{1.\frac{1}{4}}{1.\frac{1}{4}+\frac{1}{2}.\frac{3}{4}}=\frac{2}{5}$$

3. (0.73) There are two mutually exclusive cases for the event.
A = India wins the toss and wins the match
B = India losses the toss and wins the match
$\therefore$ required probability

$$=P(A)+P(B)=\frac{3}{4}\times\frac{4}{5}+\frac{1}{4}\times\frac{1}{2}=\frac{29}{40}$$

4. (1568) Let G represents drawing a green ball and R represents drawing a red ball.
So, the probability that second drawn ball is red

$$=P(G)\cdot P\left(\frac{R}{G}\right)+P(R)P\left(\frac{R}{R}\right)$$

$$=\frac{2}{7}\times\frac{6}{7}+\frac{5}{7}\times\frac{4}{7}=\frac{12+20}{49}=\frac{32}{49}=\frac{a}{b}$$

$\Rightarrow a\times b=32\times 49=1568.$

5. (0.25) $P[B/(A\cup B^c)]=\frac{P(B\cap(A\cup B^c))}{P(A\cup B^c)}$

$$=\frac{P(A\cap B)}{P(A)+P(B^c)-P(A\cap B^c)}$$

$$=\frac{P(A)-P(A\cap B^c)}{P(A)+P(B^c)-P(A\cap B^c)}=\frac{0.7-0.5}{0.8}=\frac{1}{4}$$

6. (0.78) Here $P(A)=0.4$ and $P(\overline{A})=0.6$
Probability that A does not happen at all $=(0.6)^3$
Thus required Probability $=1-(0.6)^3=0.784$.

7. (5) Let the number of independent shots required to hit the target at least once be n, then

$$1-\left(\frac{2}{3}\right)^n>\frac{5}{6}\Rightarrow\left(\frac{2}{3}\right)^n<\frac{1}{6}$$

Hence, the above inequality holds when least value of n is 5.

8. (0.14) Given : Probability of aeroplane I, scoring a target correctly i.e., $P(I)=0.3$ probability of scoring a target correctly by aeroplane II, i.e. $P(II)=0.2$

$\therefore P(\overline{I})=1-0.3=0.7$

$\therefore$ The required probability

$=P(\overline{I}\cap II)=P(\overline{I}).P(II)=0.7\times 0.2$