IOQM (2022-23)

Time: 3 hours **Max. Marks: 100**

INSTRUCTIONS

1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is STRICTLY PROHIBITED. Only ordinary pens and pencils are allowed inside the examination hall.

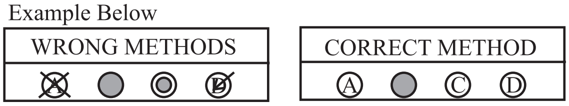
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black** or **blue ball pen**. Please DO NOT use a pencil or a gel pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

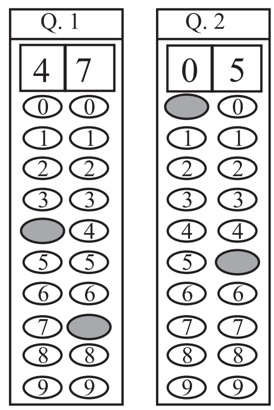
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS
1. "Think before your ink"
2. Marking should be done with Blue/ Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 8 carry 2 marks each; questions 9 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. A triangle ABC with $AC = 20$ is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding $S/20$. **[IOQM-2023]**

2. In a parallelogram $ABCD$, a point P on the segment AB is taken such that $\frac{AP}{AB} = \frac{61}{2022}$ and a point Q on the segment AD is taken such that $\frac{AQ}{AD} = \frac{61}{2065}$. If PQ intersects AC at T, find $\frac{AC}{AT}$ to the nearest integer. **[IOQM-2023]**

3. In a trapezoid $ABCD$, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if $AB : MP = 2$. **[IOQM-2023]**

4. Starting with a positive integer M written on the board, Alice plays the following game: in each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of $M + N$. **[IOQM-2023]**

5. Let m be the smallest positive integer such that $m^2 + (m + 1)^2 + ... + (m + 10)^2$ is the square of a positive integer n. Find $m + n$. **[IOQM-2023]**

6. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$. **[IOQM-2023]**

7. Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, ..., 29, 30\}$ and
$GCD(a, b) + \text{LCM}(a, b) = a + b$.

[IOQM-2023]

8. Suppose the prime numbers p and q satisfy $q^2 + 3p = 197p^2 + q$. Write $\frac{q}{p}$ as $l + \frac{m}{n}$, where l, m, n are positive integers, $m < n$ and $\text{GCD}(m, n) = 1$. Find the maximum value of $l + m + n$.

[IOQM-2023]

9. Two sides of an integer sided triangle have lengths 18 and x where $x < 100$. If there are exactly 35 possible integer values y such that 18, x, y are the sides of a non-degenerate triangle, find the number of possible integer values x can have. **[IOQM-2023]**

10. Consider the 10-digit number $M = 9876543210$. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from $M = 9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M. **[IOQM-2023]**

11. Let AB be a diameter of a circle ω and let C be a point on ω, different from A and B. The perpendicular from C intersects AB at D and ω at $E(\neq C)$. The circle with centre at C and radius CD intersects ω at P and Q. If the perimeter of the triangle PEQ is 24, find the length of the side PQ. **[IOQM-2023]**

12. Given ΔABC with $\angle B = 60^\circ$ and $\angle C = 30^\circ$, let P, Q, R be points on sides BA, AC, CB respectively such that $BPQR$ is an isosceles trapezium with $PQ \parallel BR$ and $BP = QR$. Find the minimum possible value of $\frac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon S.

[IOQM-2023]

13. Let ABC be a triangle and let D be a point on the segment BC such that $AD = BC$. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees. **[IOQM-2023]**

14. Let x, y, z be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$

$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\dfrac{x}{yz}+\dfrac{y}{zx}+\dfrac{z}{xy}=\dfrac{m}{n}$ where m, n are positive integers with $GCD(m, n) = 1$, find $m + n$.

[IOQM-2023]

15. Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression

$$\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}.$$

If $T + t$ can be expressed in the form $\dfrac{m}{n}$ where m, n are nonzero integers with $GCD(m, n) = 1$, find the value of $m + n$. **[IOQM-2023]**

16. Let a, b, c be reals satisfying

$3ab + 2 = 6b$, $3bc + 2 = 5c$, $3ca + 2 = 4a$.

Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product abc can take two values $\dfrac{r}{x} \in \mathbb{Q}$ and $\dfrac{t}{u} \in \mathbb{Q}$, in lowest form, find $r + s + t + u$. **[IOQM-2023]**

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5$, $f(10) = 5$ and $g(13) = 1$, $f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

[IOQM-2023]

18. Let m, n be natural numbers such that

$m + 3n - 5 = 2\text{LCM}(m, n) - 11\text{GCD}(m, n)$.

Find the maximum possible value of $m + n$.

[IOQM-2023]

19. Consider a string of n 1's. We wish to place some + signs in between so that the sum is 1000. For instance, if $n = 190$, one may put + signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of a. **[IOQM-2023]**

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, ..., p_n)$ of $\{1, 2, ..., n\}$, we say pl is a *landmark point* if $2 \leq l \leq n - 1$ and $(pl - 1 - pl)(pl + 1 - pl) > 0$. For example, for $n = 7$, the permutation (2, 7, 6, 4, 5, 1, 3) has four landmark points: $p_2 = 7$, $p_4 = 4$, $p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutations of $\{1, 2, ..., n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which $L(n)$ is a perfect square. **[IOQM-2023]**

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N. **[IOQM-2023]**

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

[IOQM-2023]

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ratio 1 : 2. Extend AD to E such that EB is perpendicular AB. Given that $BE = 3$, $BA = 4$, find the integer nearest to BC^2.

[IOQM-2023]

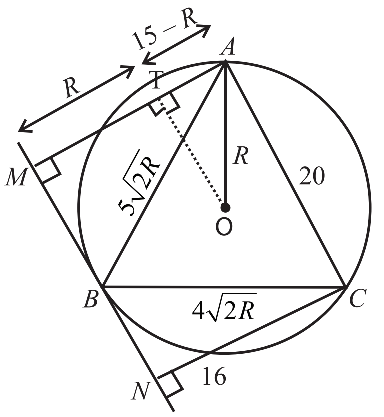
24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find $a + b$. **[IOQM-2023]**

Answers

1. (10) 2. (67) 3. (60) 4. (10) 5. (95) 6. (43) 7. (35) 8. (32) 9. (82)
10. (88) 11. (08) 12. (03) 13. (59) 14. (16) 15. (25) 16. (18) 17. (19) 18. (70)
19. (09) 20. (03) 21. (74) 22. (11) 23. (29) 24. (81)

Hints & Solutions

1. (10)



Let O be the circumcentre of ΔABC and let M, N be the foot of perpendiculars from A and C to the tangent t respectively

$$OT = \sqrt{R^2 - (25-R)^2} = 5\sqrt{2R-25} = BM$$

$$AB = \sqrt{AM^2 + BM^2} = \sqrt{25^2 + 25(2R-25)} = 5\sqrt{2R}$$

Similarly,

$$OT = 4\sqrt{2R - 16} = BN$$

$$BC = \sqrt{BN^2 + CN^2} = 4\sqrt{2R}$$

$$[ABC] = \frac{1}{2} AC \times BC \times \sin C$$

We know that,

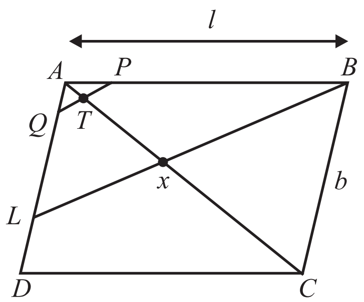
$AB = 2R \sin C$

$$\Rightarrow \sin C = \frac{5\sqrt{2R}}{2R} = \frac{5}{\sqrt{2R}}$$

$$\Rightarrow [ABC] = \frac{1}{2} \times 20 \times 4\sqrt{2R} \times \frac{5}{\sqrt{2R}} = 200 = S$$

Therefore, $\frac{S}{20} = 10$

2. (67)



Let $AB = l, AD = b$

$$\Rightarrow AP = \frac{61l}{2022};\ AQ = \frac{61b}{2065}$$

Draw a line parallel to PQ that passes through B and let it intersect AD at L Since,

$\Delta APQ \sim \Delta ABL$

$$\Rightarrow \frac{AL}{AQ} = \frac{AB}{AP} \Rightarrow AL = \frac{2022b}{2065}$$

Hence, L is a point inside the segment $\overline{AD}$

Observe that,

$\Delta AXL \sim \Delta CXB$, since $AD \parallel BC$

$$\Rightarrow \frac{AX}{CX} = \frac{AL}{BC} = \frac{2022}{2065}$$

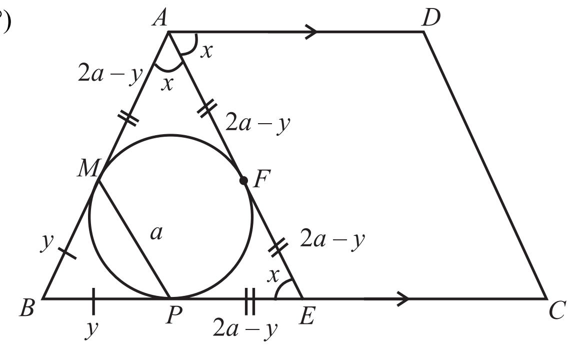
$$\Rightarrow \frac{AX}{AC} = \frac{1}{1 + \frac{CX}{AX}} = \frac{2022}{4087}$$

Since $PQ \parallel BL$,

$$\Rightarrow \frac{AT}{AX} = \frac{AP}{AB} = \frac{61}{2022}$$

$$\Rightarrow \frac{AC}{AT} = \frac{4087}{61} = 67.$$

3. (60°)



Let $AB = 2a$, $MP = a$

$\angle BAE = \angle DAE = x$ (AE is angle bisector)

$\angle AEB = \angle EAD = x$ (A.I.A.)

$\Rightarrow AB = BE = 2a$

$BM = BP$ (Length of tangents are equal)

As $\frac{BP}{BE} = \frac{BM}{BA} \Rightarrow MP \parallel AE$

So, $\Delta BMP \sim \Delta BAE$

$$\frac{BM}{BA} = \frac{MP}{AE}$$

$$\Rightarrow \frac{y}{2a} = \frac{a}{4a - 2y}$$

$\Rightarrow \ y^2 - 2ay + a^2 = 0$

$\Rightarrow \ (y - a)^2 = 0 \ \Rightarrow \ y = a$

So, ΔABE is equilateral triangle.

So, $x = 60 = \angle DAE$.

4. (10) $M \to 3M + 2 \to 9M + 8 \to 27M + 26 \to 81M + 80$

$N \to 2N + 27 \to 4N + 81 \to 8N + 189 \to 16N + 405$

$81M + 80 = 16N + 405$

$81M - 405 = 16N - 80$

$81(M - 5) = 16(N - 5)$

Above equation is valid only when $M = N = 5$

$M + N = 10$.

5. (95) $n^2 = m^2 + (m+1)^2 + ... + (m+10)^2$

We know that, $1^2 + 2^2 + ... + k^2 = \dfrac{k(k+1)(2k+1)}{6}$

$\Rightarrow \ n^2 = \dfrac{(m+10)(m+11)(2m+21)}{6} - \dfrac{(m-1)m(2m-1)}{6}$

After simplification,

$n^2 = 11(m^2 + 10m + 35) = 11((m + 5)^2 + 10)$

Let $y = m + 5$,

$n^2 = 11(y^2 + 10)$, since 11 is a prime

$\Rightarrow$ 11 divides $y^2 + 10$

$\Rightarrow \ y^2 \equiv 1 \pmod{11}$

$\Rightarrow \ y = 11s + 1, \ s \in \mathbf{N}$

Hence, $\left(\dfrac{n}{11}\right)^2 = 11s^2 \pm 2s + 1$ = Perfect square

Now substituting $s = 1, 2$ we obtain a perfect square for $s = 2$ and a '+' sign as follows,

$\left(\dfrac{n}{11}\right)^2 = 11(2)^2 + 2(2) + 1 = 7^2 \Rightarrow n = 77$

$y = 11s + 1 = 11(2) + 1 = 23 = m + 5 \ \Rightarrow \ m = 18$

$\Rightarrow \ m + n = 18 + 77 = 95$.

6. (43) $a, b \in \mathbf{N}$,

$a^3 - b^3 - ab = 25$

$\Rightarrow \ (a-b)[a^2 + b^2 + ab] - ab = 25$

$\Rightarrow \ (a-b)[(a-b)^2 + 3ab] - ab = 25$

Note that if $a - b \le 0$, then LHS < 0. So, $a - b > 0$

$(a - b)^3 + 3ab(a - b) - ab = 25$

$\Rightarrow \ (a - b)^3 + ab[3(a - b) - 1] = 25$

Observe that, $3(a - b) - 1 > 0$

$\Rightarrow \ a - b = 1$ (or) 2 else LHS > 25

Let $a - b = 2$,

$\Rightarrow \ 8 + ab(5) = 25$, but $5 \ne 8$, hence contradiction

Let $a - b = 1$,

$\Rightarrow \ 1 + ab(2) = 25 \ \Rightarrow \ ab = 12$

$\Rightarrow \ a = 4, b = 3$

Max $(a^2 + b^3) = 4^2 + 3^3 = 43$

7. (35) Let $gcd(a, b) = d$

So, $a = dp$ and $b = dq$ with $gcd(p, q) = 1$

Let $b > a$ then $q > p$

So $lcm(a, b) = aq = bp = dpq > a + b$ if $p, q > 1$, which can be verified easily.

So, one of p, q is 1 or both p, q is 1.

One of p, q is 1 implies one of a, b is multiple of another.

Thet gives us (10, 20), (10, 30), (11, 22), ... (15, 30) that is 7 pair.

If $p = q = 1$ then $a = b$ then there are 21 possible pair.

So total number of ordered pairs is $21 + 2 \times 7 = 35$

8. (32) $q^2 + 3p = 197p^2 + q$

$\Rightarrow \ q(q - 1) = p(197p - 3)$ ...(i)

Observe that prime $q > p \ \Rightarrow \ p \mid q - 1$

So, let $q = k_1 p + 1, \ k_1 \in \mathbf{N}$

Now multiply k_1^2 throughout equation (i),

$k_1^2 q(q-1) = k_1 p(197k_1 p - 3k_1)$

$= (q-1)(197q - 197 - 3k_1)$

$\Rightarrow \ k_1^2 q = 197q - 197 - 3k_1$

$\Rightarrow \ (197 - k_1^2)q = 3k_1 + 197$ ...(ii)

Hence, k_1 can take values from 1 to 14.

But $197 - k_1^2 \le \dfrac{3k_1 + 197}{2}$ (from (ii))

$\Rightarrow \ 10 \le k_1 \le 14$

Substituting these 5 values for k_1, we observe that q is an integer only for $k_1 = 14$

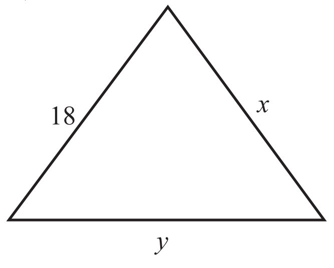
$q = \dfrac{3k_1 + 197}{197 - k_1^2} = 239 \Rightarrow q = 239,$ which is a prime.

$q = k_1 p + 1 \Rightarrow p = 17$

$\dfrac{q}{p} = \dfrac{239}{17} = 14 + \dfrac{1}{17} \Rightarrow l - 14, \ m = 1, \ n = 17$

$l + m + n = 14 + 1 + 17 = 32$.

9. (Bonus)



If we take $x = 18$ then $x - 18 < y < 18 + x$

$0 < y < 36$

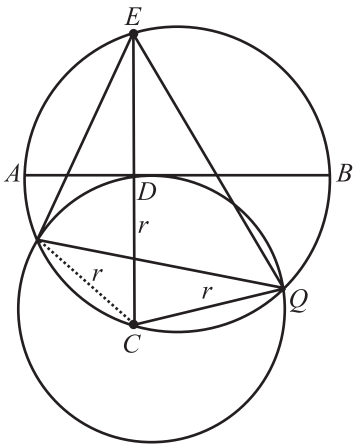
So y can take 35 integral values.
If we take $x = 19$, then $1 < y < 37$
So, y can take 35 integral value
$\Rightarrow$ If we take $x \geq 18$ we always get 35 integral value of y or in general
$y = x - 17, x - 16, ..., x + 17$
So, we can take any value of $x \geq 18$

10. (88) 9 8 7 6 5 4 3 2 1 0

Total number of pairs = 9

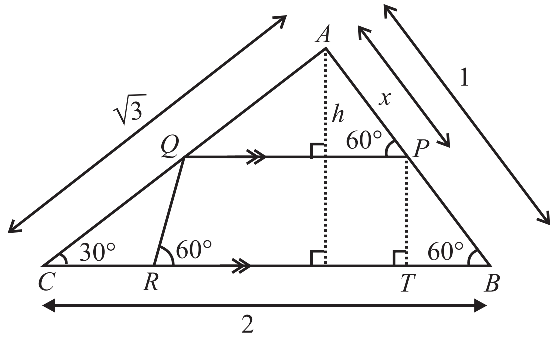
Number of pairs exchange	Equation	Number of solution
2	$x_1 + x_2 = 8$	${}^9c_1 = 9$
3	$x_1 + x_2 + x_3 = 6$	${}^8c_2 = 28$
4	$x_1 + x_2 + x_3 + x_4 = 4$	${}^7c_3 = 35$
5	$x_1 + x_2 + x_3 + x_4 + x_5 = 2$	${}^6c_4 = 15$
6	one way	1
		Total = 88

11. (08)



Let $CD = r$ = radius of smaller circle
$PCQE$ is cyclic quadrilateral.
In $PCQE$ by ptolmy's theorem :
$PE \times CQ + PC \times QE = CE \times PQ$
$r \times PE + r \times QE = 2r \times PQ$
$2PQ = PE + QE$ ...(i)
Perimeter of $\Delta PEQ = 24$
$PE + QE + PQ = 24$
$PE + QE = 24 - PQ$
From equation (i)
$2PQ = 24 - PQ$
$\Rightarrow 3PQ = 24$
$\Rightarrow PQ = 8.$

12. (Bonus)



Since, the problem asks for only ratios, we can assume $AB = 1,\ AC = \sqrt{3},\ BC = 2,$

Let $AP = x,$ due to sin values,

$AQ = \sqrt{3}x,\ PQ = 2x$

Let h be the height of ΔAQP on QP,

$$\frac{1}{2}h(PQ) = \frac{1}{2}(\sqrt{3}x)x \Rightarrow h = \frac{\sqrt{3}x}{2}$$

Height of trapezium = height of ΔABC on $BC - h$

$$= \frac{\sqrt{3}}{2}(1-x)$$

Let T be feet of $\perp$ from P to BC,

$$\Rightarrow \quad BT = BP\ \cos 60 = \frac{1-x}{2}$$

$\Rightarrow$ $BR = PQ + 2BT$ (since it is isosceles trapezium)
$= 2x + 1 - x = 1 + x$

$$\frac{2[ABC]}{[BPQR]} = \frac{2 \times \frac{1}{2} \times \sqrt{3} \times 1}{\frac{1}{2} \times \text{height} \times \text{sum of } ||^{el} \text{ sides}}$$

$$= \frac{2\sqrt{3}}{\frac{\sqrt{3}}{2}(1-x)(1+3x)}$$

$$\Rightarrow \quad \frac{2[ABC]}{[BPQR]} = \frac{4}{(1-x)(1+3x)} = \frac{4}{\frac{4}{3} - \left(\sqrt{3}x - \frac{1}{\sqrt{3}}\right)^2}$$

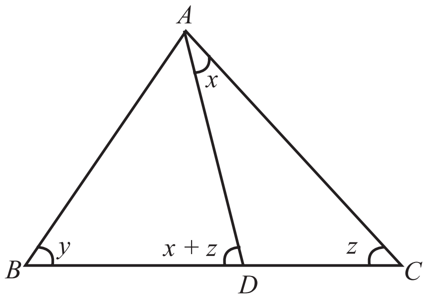
Note that this expression has no upper bound and hence no maximum value. Thus, let us change this question and find the minimum value. It happens when,

$$\sqrt{3}x - \frac{1}{\sqrt{3}} = 0 \Rightarrow x = \frac{1}{3}$$

$$\Rightarrow \quad \min\left(\frac{2[ABC]}{[BPQR]}\right) = \frac{4}{4/3} = 3$$

This question is solvable only if P, Q, R lie inside the sides of the triangle.

13. (59)



In ΔABD, by sine rule:

$$\frac{AD}{AB} = \frac{\sin y}{\sin(x+z)}$$

$\because$ x, y, z are in A.P. so $x + z = 2y$

$$\frac{AD}{AB} = \frac{\sin y}{\sin(x+z)} = \frac{\sin y}{\sin 2y}$$

In ΔABC by sine rule

$$\frac{BC}{AB} = \frac{\sin[180-(y+z)]}{\sin z} = \frac{\sin(y+z)}{\sin z}$$

$\because$ $AD = BC$

$\therefore$ $\dfrac{\sin y}{\sin 2y} = \dfrac{\sin(y+z)}{\sin z}$

$\Rightarrow$ $\dfrac{\sin y}{2\sin y \cos y} = \dfrac{\sin(y+z)}{\sin z}$

$\Rightarrow$ $\sin z = 2\sin(y+z)\cos y$

$\Rightarrow$ $\sin z = \sin(2y+z) + \sin z$

$\Rightarrow$ $\sin(2y+z) = 0$

$\Rightarrow$ $2y + z = 180°$

$\because$ $y < z$

$\Rightarrow$ $2y + y < 180$

$\Rightarrow$ $y < 60$ largest value of $\angle ABC = 59°$

14. (16) $\left(\dfrac{x}{y+z}+\dfrac{y}{z+x}+\dfrac{z}{x+y}\right)(x+y+z)$

$$= \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}+(x+y+z)$$

$\Rightarrow$ $9(x+y+z) = 64 + (x+y+z)$

$\Rightarrow$ $x + y + z = 8$ ...(i)

Similarly note that,

$$\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)(x+y+z)$$

$$= \frac{x^3}{y+z}+\frac{y^3}{z+x}+\frac{z^3}{x+y}+(x^2+y^2+z^2)$$

$\Rightarrow$ $64(8) = 488 + x^2 + y^2 + z^2$

$\Rightarrow$ $x^2 + y^2 + z^2 = 24$ ...(ii)

From (i) and (ii),

$$xy + yz + zx = \frac{(x+y+z)^2-(x^2+y^2+z^2)}{2} = 20 \quad \text{...(iii)}$$

Given,

$$\frac{x}{y+z}+1+\frac{y}{z+x}+1+\frac{z}{x+y}+1 = 9+3 = 12$$

$\Rightarrow$ $\dfrac{1}{y+z}+\dfrac{1}{z+x}+\dfrac{1}{x+y} = \dfrac{12}{x+y+z} = \dfrac{3}{2}$ (from (i)) ...(iv)

$\Rightarrow$ $\dfrac{1}{y+z}+\dfrac{1}{z+x}+\dfrac{1}{x+y} = \dfrac{12}{x+y+z} = \dfrac{3}{2}$

Observe that,

$$x^2+y^2+z^2 = \left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)(xy+yz+zx)$$
$$-xyz\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)$$

From (ii), (iii), (iv)

$$24 = 9(20) - \frac{3xyz}{2}$$

$\Rightarrow$ $xyz = 104$ ...(v)

Divide (ii) by (v),

$$\frac{x^2+y^2+z^2}{xyz} = \frac{24}{104}$$

$\Rightarrow$ $\dfrac{x}{yz}+\dfrac{y}{zx}+\dfrac{z}{xy} = \dfrac{3}{13}$

$\Rightarrow$ $m = 3, n = 13$

$\Rightarrow$ $m + n = 16$

15. (25) $(x+y)^2 = (x-y)^2 + 4xy$
$= (x-y)^2 + 4$

So,

$$K = \frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2} = \frac{(x-y)^2-(x-y)+2}{(x-y)^2+(x-y)+2}$$

Let $x - y = a$

$$K = \frac{a^2-a+2}{a^2+a+2}$$

$(K+1)a^2 + (K+1)a + 2(K-1) = 0$

as a is real, so $D \geq 0$

$(K+1)^2 - 4(K-1)2(K-1) \geq 0$

$\Rightarrow$ $(K+1)^2 - 8(K-1)^2 \geq 0$

$\Rightarrow$ $[K(1+2\sqrt{2})+(1-2\sqrt{2})][K(2\sqrt{2}-1)-1-2\sqrt{2}] \leq 0$

$\Rightarrow$ $T = \dfrac{2\sqrt{2}+1}{2\sqrt{2}-1}, \ t = \dfrac{2\sqrt{2}-1}{2\sqrt{2}+1}$

$\Rightarrow \ T + t = \dfrac{2\sqrt{2}+1}{2\sqrt{2}-1} + \dfrac{2\sqrt{2}-1}{2\sqrt{2}+1}$

$= \dfrac{8+1-8\sqrt{2}+8+1+8\sqrt{2}}{8-1}$

$= \dfrac{18}{7} = \dfrac{m}{n}$

$\Rightarrow \ m + n = 18 + 7 = 25$

16. (18) $3ab + 2 = 6b$

$\Rightarrow \ a = \dfrac{6b-2}{3b}$

$\Rightarrow \ 3bc + 2 = 5c$

$\Rightarrow \ c = \dfrac{2}{5-3b}$

$\Rightarrow \ 3ca + 2 = 4a$

$\Rightarrow \ 3\left(\dfrac{6b-2}{3b}\right)\left(\dfrac{2}{5-3b}\right) + 2 = 4\left(\dfrac{6b-2}{3b}\right)$

$\Rightarrow \ 27b^2 - 39b + 14 = 0$

$\Rightarrow \ 27b^2 - 18b - 21b + 14 = 0$

$\Rightarrow \ (3b-2)(9b-7) = 0$

$\Rightarrow \ b = \dfrac{2}{3}, \dfrac{7}{9}$

When $b = \dfrac{7}{9}$, $a = \dfrac{8}{7}$ and $c = \dfrac{3}{4}$

$abc = \dfrac{7}{9} \times \dfrac{8}{7} \times \dfrac{3}{4} = \dfrac{2}{3} = \dfrac{r}{s}$

When $b = \dfrac{2}{3}$, $a = 1$ and $c = \dfrac{2}{3}$

$abc = \dfrac{2}{3} \times 1 \times \dfrac{2}{3} = \dfrac{4}{9} = \dfrac{t}{u}$

$\therefore \ r + s + t + u = 2 + 3 + 4 + 9 = 18$

17. (19) $f(n) = n - g(n)$

If n is even

$n = 8$, $g(8) = 4$, $f(8) = 4$

$n = 50$, $g(50) = 25$, $f(50) = 25$

$n = 12$, $g(12) = 6$, $f(12) = 6$

hence if $n = x$, where x is even, then $f(n) = \dfrac{x}{2}$

18. (70) Let $gcd(m, n) = d$

$\Rightarrow \ m = dp,\ n = dq,$ where $gcd(p, q) = 1$

$mn = gcd(m, n) \times lcm(m, n) \Rightarrow lcm(m, n) = dpq$

Substituting them in the equation,

$dp + 3dq - 5 = 2dpq - 11d$

$\Rightarrow \ d(p + 3q - 2pq + 11) = 5$

$\Rightarrow \ d = 1$ (or) 5

Case 1 :

$d = 1 \Rightarrow p + 3q - 2pq + 11 = 5$

By rearranging and taking the terms common,

$(2q - 1)(2p - 3) = 15$

Observe that $p, q \in \mathbb{N} \Rightarrow 2q - 1 > 0 \Rightarrow 2p - 3 > 0$ and so, we consider only positive divisors of 15.

$2q - 1 = 1;\ 2p - 3 = 15$

$2q - 1 = 3;\ 2p - 3 = 5$

$2q - 1 = 5;\ 2p - 3 = 3$

$2q - 1 = 15;\ 2p - 3 = 1$

$\Rightarrow \ (p, q) = (9, 1); (4, 2); (3, 3); (2, 8)$

But, the last 3 solutions are invalid, as the $gcd(p, q)$ is 1.

$\Rightarrow \ (p, q) = (9, 1)$

$\Rightarrow \ (m, n) = (dp, dq) = (9, 1)$

Case 2 :

$d = 5 \Rightarrow p + 3q - 2pq + 11 = 1$

$\Rightarrow \ (2q - 1)(2p - 3) = 23$

$\Rightarrow \ 2q - 1 = 1;\ 2p - 3 = 23$

$\Rightarrow \ 2q - 1 = 23;\ 2p - 3 = 1$

$\Rightarrow \ (p, q) = (13, 1); (2, 12)$

The latter solution is invalid.

$\Rightarrow \ (p, q) = (13, 1)$

$\Rightarrow \ (m, n) = (dp, dp) = (65, 5)$

Hence, max $(m + n) = 65 + 5 = 70$

19. (09) Observe that after inserting the '+' signs, the numbers that one would get are possibly 1, 11, 111. Rest all the numbers are greater than 1000. After splitting let the numbers of 1's, 11's, and 111's be k_1, k_2, k_3 respectively.

$\Rightarrow \ k_1 + 11k_2 + 111k_3 = 1000;\ k_1 + 2k_2 + 3k_3 = n$

Now we just need to find the number of possible n's such that $k_1 + 11k_2 + 111k_3 = 1000$ and we need not find the number of solutions of k_1, k_2, k_3, since $k_1, k_2, k_3 \in \mathbf{N}_0$

$n = 1000 - 9k_2 - 108k_3$ gives the no. of values of n for $k_2, k_3 \geq 0$ and also we need

$k_1 \geq 0 \Rightarrow n - (3k_3 + 2k_2) \geq 0$

For $k_2, k_3 \geq 0$

The final two equations are,

$n = 1000 - 9(12k_3 + k_2);\ n - (3k_3 + 2k_2) \geq 0$

Now $12k_3 + k_2$ can take all the values from 0 to 111 as there always exist a quotient and remainder when divide by 12.

$\Rightarrow$ n can take 112 values, but out of these some values will have $n - (3k_3 + 2k_2) < 0$ and we need to eliminate them.

20. (03) Let us assume we are setting one peak then it must be 'n' and rest of the number can be arranged left right of it in 2^{n-1} ways!!

Removing strictly increasing (decreasing) arrangement we set $2^{n-1} - 2$ ways.

If we get n valley then it must be '1' and again we get $2^{n-1} - 2$ ways.

Similarly if we get a valley, then it must be 1 and again we get $2^{n-1} - 2$ ways.

Total ways $= 2^{n-1} + 2^{n-1} - 4$

$= 2^n - 4 = 4(2^{n-2} - 1)$

$L(n) = 4(2^{n-2} - 1)$

$L(n)$ is perfect square

$\Rightarrow$ $2^{n-2} - 1$ must be square.

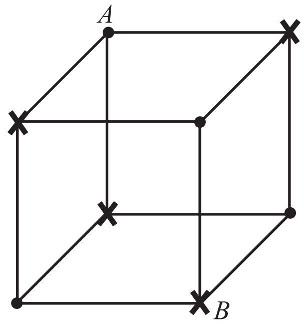
$\Rightarrow$ $2^{n-2} - 1 \equiv 1 \pmod 8$

$\Rightarrow$ $2^{n-2} \equiv 2 \pmod 8$

$\Rightarrow$ $2^{n-3} \equiv 1 \pmod 4$

$\Rightarrow$ $n = 3$

21. (74)



In the cube, mark the alternate vertices with '•' and '×' as shown in the diagram. Note that for any move, the ant alternates between a dot and a cross in each move. Suppose the ant starts from the vertex A and let the diagonally opposite vertex of A be B.

According to the problem, we need to find the number of ways in which the ant covers 6 edges (not necessarily distinct) and come back to A after its 6[th] move.

Since, the ant alternates the dot and the corss for any possible move of the ant, it should be at a cross after its 5[th] move. Note that all the cross expect B are adjacent to A, and hence it can reach A. So, we need to subtract the no. of ways to reach B from the total number of ways for the 5 moves.

There are a total of 3 paths to choose for the ant from each vertex and hence total of 3^5 ways for the 5 moves.

$N = 3^5$ – (no. of ways to reach B in 5 moves)

After 4 moves, the ant must be at 'dot' and all the dots except A are adjacent to B and hence it can move to B. Hence,

No. of ways to reach B in 5 moves

$= 3^4$ – (no. of ways to reach A in 4 moves)

Similarly arguing we get,

No. of ways to reach A in 4 moves

$= 3^3$ – (no. of ways to reach B in 3 moves)

So we get,

$N = 3^5 - (3^4 - (3^3 - (3^2 - (3 -$ (no. of ways to reach B in 1 move)))))

Since A and B are not adjacent and the ant starts from A, there is no way for the ant to reach B in 1 move.

$N = 3^5 - 3^4 + 3^3 - 3^2 + 3 = 183$

Sum of squares of digits of $N = 1^2 + 8^2 + 3^2 = 74$

22. (11) We note that each run of 1s is in friendly binary sequence has to have length two or more for $n \geq 5$, we partition the sequence counted by F_n into three cases.

Case I : The first run of 1s has length atleast three. In this case, removing one of the 1s in the 1[st] run leaves a friendly sequence of length $n - 1$.

Conversely every friendly sequence of length $n - 1$ can be extended to a friendly sequence of length n by inserting a 1 into the 1[st] run of 1s.

For Eg. : 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0 can be mapped to

0, 1, 1, 1, 1, 0, 0, 1, 1, 0.

so there are F_{n-1} sequences in this case.

Case 2 : The 1[st] run of 1s has length two and the 1[st] term in the sequence is 1. In this case, the sequence begins with 1, 1, 0 and what follows is any one of the F_{n-3} friendly sequences of length $n - 3$.

Case 3 : The first run of 1s has length two and the 1[st] term in the sequence is 0. In this case, the sequence begins 0, 1, 1, 0 and what follows is any one of the F_{n-4} friendly sequences of length $n - 4$. So from case 1, 2 and 3, we conclude that $F_n = F_{n-1} + F_{n-3} + F_{n-4}$.

For $n \geq 5$, by checking small cases, we verify that $F_1 = 0, F_2 = 1, F_3 = 3, F_4 = 4$.

Now by above recurrence,

$F_6 = F_5 + F_3 + F_2 = 5 + 3 + 1 = 9$

$F_7 = F_6 + F_4 + F_3 = 9 + 4 + 3 = 16$

$F_8 = F_7 + F_5 + F_4 = 16 + 5 + 4 = 25$

$F_9 = F_8 + F_6 + F_5 = 25 + 9 + 5 = 39$

$F_{10} = F_9 + F_7 + F_6 = 39 + 16 + 9 = 64$

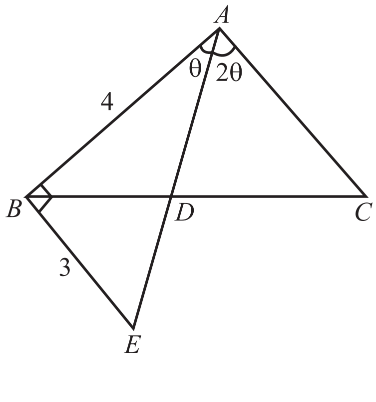
$F_{11} = F_{10} + F_8 + F_7 = 64 + 25 + 16 = 105 > 100$

hence $n = 11$

23. (29) Let $\angle BAE = \theta \Rightarrow \angle CAE = 2\theta$

Given that $\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5} \Rightarrow 30° < \theta < 45°$

Hence, $\angle BAC = 3\theta > 90°$ is obtuse and $\angle ABC$ will be acute, so the point E lies outside ΔABC as shown in the figure.



$$\frac{[ABD]}{[ACD]} = \frac{-\times h \times BD}{-\times h \times CD} =$$

$$= \frac{\frac{1}{2}(AB)(AD)\sin\theta}{\frac{1}{2}(AC)(AD)\sin 2\theta} \qquad \text{(Since } BD = CD\text{)}$$

$$\Rightarrow \quad AC = \frac{4}{2\cos\theta} = \frac{4}{2\left(\frac{4}{5}\right)} \Rightarrow AC = \frac{5}{2}$$

$$\Rightarrow \quad \cos 3\theta = \cos\angle BAC = 4\cos^3\theta - 3\cos\theta = -\frac{44}{125}$$

By cosine law in ΔABC for the $\angle BAC$,

$$\cos 3\theta = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC} = \frac{4^2 + \left(\frac{5}{2}\right)^2 - BC^2}{2 \times 4 \times \frac{5}{2}} = -\frac{44}{125}$$

$$\Rightarrow \quad BC^2 = 16 + \frac{25}{4} + \frac{880}{125} = 29 + \frac{1}{4} + \frac{1}{25}$$

$\Rightarrow$ $BC^2 = 29.29$ units^2

Therefore, integer nearest to $BC^2 = 29$.

24. (81) The number of ball in the boxes are in the form $6k + r$; $r = \{0, 1, 2, 3, 4, 5\}$

Let the number of balls in the boxes be $6k + a$, $6k + b$, $6k + c$, $6k + d$, respectively, where $a, b, c, d \in r$ and are different from each other.

So, $52 - (a + b + c + d) = 6n$

That gives $a + b + c + d = 10$ and no other values are possible to obtain distinct remainders

$\Rightarrow$ $n = 7$

Case 1 :

So, the remainders are 1, 2, 3, 4 and can be arranged 4! ways.

Now there are 7 identical collection of 6 balls each which we have to divide it into 4 distinct boxes.

Using stars and bars method we get that,

there are ${}^{10}C_3 = 120$ ways to distribute them in those distinct boxes.

So total number of ways is $120 \times 24 = 2880$.

Case 2 :

The remainders can also be 0, 2, 3, 5 (or) 0, 1, 4, 5 as it sums up to 10 as well.

There are no more remainders which add up to 10.

Hence we just get 3×2880 in total.

But now, there could be some empty boxes due to that 0 remainder and hence we need to subtract that out!, which is

$4(2(3!) \times {}^9C_2) = 1728$

$N = 3 \times 2880 - 1728 = 6912$

$a = 69$; $b = 12$

Hence, $a + b = 81$