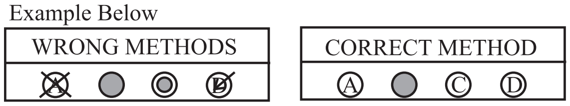
IOQM (2023-24)

Time: 3 hours **Max. Marks: 100**

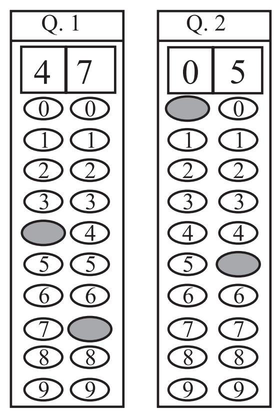
INSTRUCTIONS

1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is STRICTLY PROHIBITED. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black** or **blue ball pen**. Please DO NOT use a pencil or a gel pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS
1. "Think before your ink"
2. Marking should be done with Blue/ Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 to 30 carry 5 marks each.
8. All questions are compulsory.
9. There are no negative marks.
10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
11. After the exam, you may take away the candidate's copy of the OMR sheet.
12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
13. You may take away the question paper after the examination.

Note:

1. $\mathbb{N}$ denotes the set of all natural numbers, 1, 2, 3, ...
2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. For example, $\sqrt{4} = +2$
3. Unless otherwise specified, all numbers are written in base 10.

Question:

1. Let n be a positive integer such that $1 \le n \le 1000$. Let M_n be the number of integers in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2} ..., \sqrt{4n+1000}\}$. Let $a = \max\{M_n : 1 \le n \le 1000\}$, and $b = \min\{M_n : 1 \le n \le 1000\}$. Find $a - b$.
2. Find the number of elements in the set
$\{(a, b) \in \mathbb{N} : 2 \le a, b \le 2023, \log_a(b) + 6 \log_b(a) = 5\}$
3. Let α and β be positive integers such that
$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$.
Find the smallest possible value of β.
4. Let x, y be positive integers such that
$x^4 = (x - 1)(y^3 - 23) - 1$.
Find the maximum possible value of $x + y$.
5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480 , find the area of triangle GYZ.
6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.
7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.
8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.
9. Find the number of triples (a, b, c) of positive integers such that
   (a) ab is a prime;
   (b) bc is a product of two primes;
   (c) abc is not divisible by square of any prime and
   (d) $abc \le 30$
10. The sequence $\langle a_n \rangle_{n \ge 0}$ is defined by $a_0 = 1$, $a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \ge 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.
11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.
12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.
13. The ex-radii of a triangle are $10\frac{1}{2}$, 12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$
14. Let ABC be a triangle in the xy-plane, where B is at the origin (0, 0) . Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF . Find the length GK.
15. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2 . Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n , find the value of $m + n$.
16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.
17. Consider the set
$S = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$
where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.
18. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal $d \in F$ intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F.
19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set $A = \{n \in \mathbb{N} : P(n)$ is non-zero, square free and $S(n)$ is a proper divisor of $P(n)\}$:
Find the maximum possible number of digits of the numbers in A.

20. For any finite non-empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that $\max(A) \times |B| = 12$, and $|A| \times \max(B) = 11$ and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

21. For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n}(i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$.

25. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations 1, 2, 2^2, 2^3, ... Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27. A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a + c = b + d$. Let S be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in S.

28. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \leq n \leq 100$, consider $n - 1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them is 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves

29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \ldots\ldots a_k$ for some positive integers $a_1, a_2, \ldots, a_k$, where $k > 1$ and $a_1 \geq a_2 \geq \ldots\ldots \geq a_k$. (For example 6 is beautiful since $6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, so uniqueness is lost.) Find the largest beautiful number less than 100.

30. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

Answers

1. (22)	2. (54)	3. (23)	4. (07)	5. (10)	6. (16)	7. (48)	8. (59)	9. (17)	10. (51)
11. (14)	12. (18)	13. (58)	14. (40)	15. (03)	16. (94)	17. (66)	18. (71)	19. (92)	20. (43)
21. (15)	22. (77)	23. (18)	24. (31)	25. (30)	26. (19)	27. (91)	28. (67)	29. (95)	30. (18)

Hints & Solutions

1. (22) $\{\sqrt{4n+1}, \sqrt{4n+2}, ...\sqrt{4n+1000}\}$

$n = 1$

$x_n = [\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{9}, \sqrt{10}, ...\sqrt{1004}]$

$n = 2$

$x_2 = [\sqrt{9}, \sqrt{10}, \sqrt{11}, ...\sqrt{1008}]$

$x_3 = [\sqrt{13}, \sqrt{14}, \sqrt{15}, ...\sqrt{1012}]$

⋮

⋮

$x_{1000} = \left\{\sqrt{4001}, \sqrt{4002}, \sqrt{4003}, ...\sqrt{5000}\right\}$

M_n is max. when $n = 1$ and $n = 2$

$a = \max.(M_n) = 29$

Min (M_n) is when $n = 1000$

$b = \min.(M_n) = 7$

$a - b = 29 - 7 = 22$

2. (54) $\log_a b = t$

$t + \frac{6}{t} = 5$

$\Rightarrow t^2 - 5t + 6 = 0 \quad \Rightarrow t = 2 \text{ or } 3$

$\log_a b = 2 \text{ or } 3 \quad \Rightarrow b = a^2 \text{ or } a^3$

$a^2 \le 2023$ and $a^3 \le 2023$

$\Rightarrow a \le 44.97$ and $a \le 12.64$

$\Rightarrow a \in \{2, ... 44\}$ and $a \in \{2, ... 12)$

$\Rightarrow (44 - 2 + 1) + (12 - 2 + 1)$

$\Rightarrow 43 + 11 = 54$

3. (23) $\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$

$16\beta < 37\alpha \quad ; \quad 16\alpha < 7\beta$

$\beta < \frac{37\alpha}{16} \quad ; \quad \beta > \frac{16\alpha}{7}$

$\frac{16\alpha}{7} < \beta < \frac{37\alpha}{16}$

For $\alpha = 1, 2, 3 ..... 9, \beta \notin I^+$

At $\alpha = 10$

$22.8571 < \beta < 23.125$

$\beta = 23$

4. (07) $\sqrt{4n+r} = k \in I$

$\Rightarrow 4n + r = k^2$

$\frac{(x^4+1)}{(x-1)} = y^3 - 23, \quad x, y \in I$

Since, $x \ne 1 \Rightarrow x \ge 2; \quad y^3 - 23 \in I$

$\Rightarrow \frac{x^4+1}{x-1} \in I$

Let, $x - 1 = p$

$\Rightarrow \frac{(1+p)^4+1}{p}$

$\Rightarrow \frac{(a_4p^4 + a_3p^3 + ... a_1p + 2)}{p}$

$\Rightarrow \frac{2}{p} \Rightarrow p$ divides 2

$\Rightarrow p = \{-2, -1, 1, 2\}$

$\Rightarrow x \in \{-1, 0, 2, 3\}$

But $x \ge 2 \Rightarrow x = 2$ or 3

If $x = 2$

$2^4 = 1 \times (y^3 - 23) - 1$

$\Rightarrow (17 + 23) = y^3$

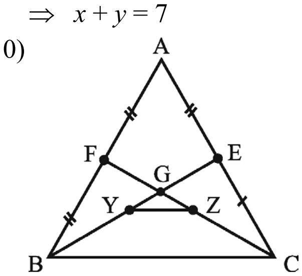
$\Rightarrow y \notin I$

If $x = 3$

$(3^4 + 1) = 2(y^3 - 23) = y = 4$

$\Rightarrow x + y = 7$

5. (10)



Let $BE = 3K$, $\quad CF = 3M$

$BY = \frac{3}{2}K$, $\quad CZ = \frac{3}{2}M$,

$YG = \frac{1}{2}K$, $\quad ZG = \frac{1}{2}M$,

$GE = K$, $\quad FG = M$

ar $(\Delta ABC) = 480$

ar $(\Delta BEC) = \frac{1}{2}$ ar $(\Delta ABC) = \frac{1}{2} \times 480 = 240$

ar $(\Delta BGC) = \frac{2}{3}$ ar $(\Delta BEC) = \frac{2}{3} \times 240 = 2 \times 80 = 160$

In ΔBGC,

$\frac{GY}{YB} = \frac{GZ}{ZC} = \frac{1}{3}$

$$\frac{ar(\Delta GYZ)}{ar(\Delta BGC)}=\left(\frac{1}{4}\right)^2$$

$$\frac{ar(\Delta GYZ)}{160}=\frac{1}{16}$$

$$ar(\Delta GYZ)=\frac{160}{16}=10$$

6. (16) Let number of sides of polygon be P.

$\therefore \quad \frac{(P-2)\cdot 180^\circ}{P}=n^\circ$

then P (180 – n) = 360

$\therefore \quad P=\frac{360}{180-n}$

But $n = 2k$, then $P=\frac{180}{90-k}$

Hence, possible values of $(90 – k)$ are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45 and 60.

$\therefore$ 16 such polygons are possible.

7. (48) Arrangement of 3, 4, 5, 6 can be done in 3! ways = 6 ways (using circular permutation)

Colouring can be done in 2 × 2 × 2 = 8 ways

$\Rightarrow$ Total design are 8 × 6 = 48 ways.

8. (59) **Case I:** If we use only dominoes.

For $2 \times n$ rectangle, we get

Recursion formula as $F(n) = F(n – 1) + F(n – 2)$

where $F(1) = 1$, $F(2) = 2$, $F(3) = 3$, $F(4) = 5$, $F(5) = 8$, $F(6) = 13$, $F(7) = 21$

Case II: When 2 × 2 tile is used.

$2 \times (F(5) + F(1) \times F(4) + F(2) \times F(3))$

$2 \times (8 + 1 \times 5 + 2 \times 3) = 38$

$\Rightarrow$ Total = 21 + 38 = 59

9. (17) $abc \le 30$ and abc is not divisible by 4, 9, 25.

So, abc can take values :

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29 and 30

$\therefore$ ab is prime.

Case I : When $a = 1$ and b is prime number.

$a = 1$,	$b = 2$,	$c = 3, 5, 7, 11, 13$
	$b = 3$,	$c = 2, 5, 7$
	$b = 5$,	$c = 2, 3$
	$b = 7$,	$c = 2, 3$
	$b = 11$,	$c = 2$
	$b = 13$,	$c = 2$

There are total 14 triples of (a, b, c)

Case 2 : When $b = 1$ and a is prime number.

$b = 1$,	$a = 2$	$c = 15$
	$a = 3$	$c = 10$
	$a = 5$	$c = 6$

There are 3 triples of (a, b, c).

From case I and case II, total 17 triples of (a, b, c) are possible.

10. (51) The sequence $\{a_n\}$, $n \ge 0$ and $a_0 = 1$, $a_1 = -4$ and given that $a_{n+2} = -4a_{n+1} - 7a_n$ ...... (i)

Now, $a_n^2 - a_{n+1}\cdot a_{n-1} = a_n^2 - (4a_n - 7a_{n-1})a_{n-1}$

$= a_n^2 + 4a_n . a_{n-1} + 7a_{n-1}^2$

$= -a_n(7a_{n-2}) + 7a_{n-1}^2$

$= 7\ (a_{n-1}^2 - a_n \cdot a_{n-2})$

$= 7^2\ (a_{n-1}^2 - a_{n-1} \cdot a_{n-3})$

....

$\therefore \quad a_n^2 - a_{n+1}a_{n-1} = 7^{n-1}(a_1^2 - a_0 a_2)$

$= 7^{n-1} = (16 – 1.9) = 7^n$

$\therefore \quad a_{50}^2 - a_{49}.a_{51} = 7^{50}$

$\therefore$ Number of positive integer divisors of 7^{50} $= 50 + 1 = 51$

11. (14) $m^2 = 4n^2 - 5n + 16$

$16m^2 = 64n^2 - 80n + 256$

$16m^2 = (8n-5)^2 + 231$

$(4m)^2 - (8n-5)^2 = 7\times 11\times 3$

$(4m+8n-5)\ (4m-8n+5) = 7\times 11\times 3$

By property and for max. of 'm'

$4m + 8m – 5 = 1$

$4m – 8m + 5 = 231$

$8m \quad = 232$

$m = 29$

So, n = not integer $8n = – 110$

So, $4m – 8m – 5 = 77$

$4m + 8m + 5 = 3$

$8m \quad = 80 \quad \Rightarrow m = 10;\ n = – 4$

$\therefore$ $|m – n| = |10 – (– 4)| = 14$

12. (18) $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$

Only possible if

$p_1 + p_2 + p_3 = 0$

$\Rightarrow 36 + 14a + 6b + 3c = 0$

$\downarrow$

Not possible to calculate.

or $p_1 = p_2 = p_3$

or $a + b + c + 1 = 8 + 4a + 2b + c = 27 + 9a + 3b + c$

$\downarrow$

Now, $3a + b = – 7$ and $5a + b = – 19$

$2a = -12$

$a = -6$

$b = 11$

Now, $p_2 + 2p_1 - 3p_0$

$\Rightarrow\ 6a + 4b + 10$

$\Rightarrow\ 6 \times (-6) + 4 \times 11 + 10$

$= -36 + 44 = 0$

$= -36 + 54 = 18$

13. (58) $a + b + c = p,\ ab + bc + ca = q$

$abc = r$

$$r_1 = \sqrt{\frac{s(s-b)(s-c)}{s-a}} = \frac{21}{2}$$

$$r_2 = \sqrt{\frac{s(s-a)(s-c)}{s-b}} = 12$$

$$r_3 = \sqrt{\frac{s(s-a)(s-b)}{s-c}} = 14$$

Now, $p + q + r = a + b + c + ab + bc + ca + abc$

$= (a + 1)(b + 1)(c + 1) - 1$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} = \frac{s}{\Delta} = \frac{p}{2\Delta}$$

$$\Rightarrow \frac{2}{21} + \frac{1}{12} + \frac{1}{14} = \frac{p}{2\Delta} = \frac{8+7+6}{84} = \frac{21}{84} = \frac{1}{4}$$

$$\Rightarrow r_1r_2 + r_2r_3 + r_3r_1 = \left(\frac{p}{2}\right)^2$$

$$\Rightarrow 126 + 168 + 147 = \frac{p^2}{4}$$

$\Rightarrow\ 441 \times 4 = p^2$

$\Rightarrow\ p = 2 \times 21 = 42$

$\Rightarrow \dfrac{42}{2\Delta} = \dfrac{1}{4} \qquad \Rightarrow \Delta = 84$

$\dfrac{\Delta}{s-a} = \dfrac{21}{2} \qquad \Rightarrow \dfrac{84}{21} \times 2 = s - a = 8$

$\dfrac{\Delta}{s-b} = 12 \qquad \Rightarrow s - b = 7$

$\dfrac{\Delta}{s-c} = 14 \qquad \Rightarrow s - c = 6$

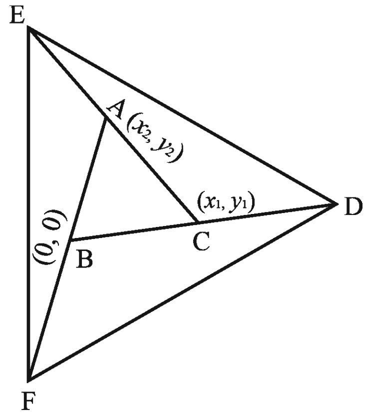
$s = 21,\ a = 13,\ b = 14,\ c = 15$

$\Rightarrow\ (a + 1)(b + 1)(c + 1) = 14 \times 15 \times 16$

$\Rightarrow\ \sqrt{(a+1)(b+1)(c+1) - 1} = 57.95$

$\Rightarrow\ 58$

14. (40)



$\because$ BC : CD = 1 : 1 hence D = $(2x_1, 2y_1)$

CA : AE = 1 : 2 hence E = $(3x_2 - 2x_1, 3y_2 - 2y_1)$

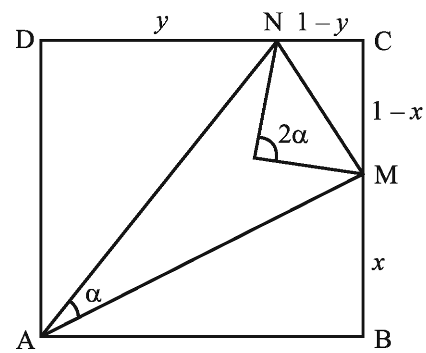
AB : BF = 1 : 3 hence F = $(-3x_2, -3y_2)$

$\therefore$ Centroid of ΔDEF = K = (0, 0)

Centroid of ΔABC = G = (32, 24)

$\therefore$ GK = $\sqrt{32^2 + 24^2} = 40$

15. (03)



$\because$ OA = Circumradius of ΔAMN

$$= \frac{MN}{2\sin\alpha}$$

OP = Circumradius of ΔOMN

$$= \frac{MN}{2\sin 2\alpha}$$

So, $\left(\dfrac{OP}{OA}\right)^2 = \left(\dfrac{1}{2\cos\alpha}\right)^2$

Perimeter of ΔMCN = 2

$= (1 - x) + (1 - y) + \text{MN}$

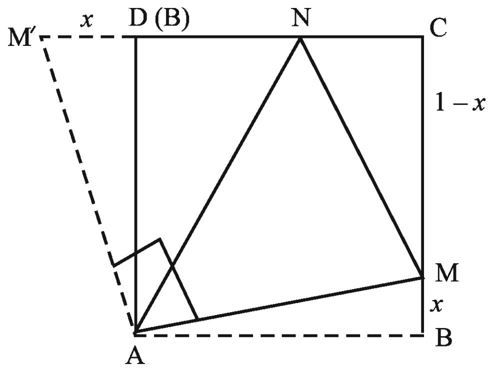
$\Rightarrow$ MN $= x + y$

Now, rotate ΔABM about A so that AB overlaps with AD (by 90°)

Clearly, ΔAMN = ΔAM'N

So, $2\alpha = 90°$

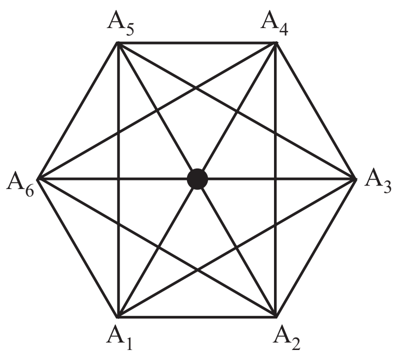
$\therefore\ \alpha = 45°$



Hence, $\left(\dfrac{OP}{OA}\right)^2 = \dfrac{1}{2} = \dfrac{m}{n}$

$\therefore \quad m + n = 3$

16. (94)



Number of ways such that atleast one side of $\Delta A_2A_4A_6$ is red = ${}^3C_1 \times 2^2 - {}^3C_2 \times 2 + {}^3C_3 2^0 = 7$

Number of ways such that atleast one side of $\Delta A_1A_3A_5$ is red = 7

Number of ways to colour diagonals A_1A_4, A_2A_5, $A_3A_6 = 2^3 = 8$

$\therefore$ Required number = 8 × 7 × 7 = 392 = N

$\therefore$ Sum of square of digits = $3^2 + 9^2 + 2^2 = 94$

17. (66) $d = 4$ sum = $4 \times {}^{95}C_1 \times {}^3C_3$

$d = 5$ sum = $5 \times {}^{94}C_1 \times {}^4C_3$

$d = 6$ sum = $6 \times {}^{93}C_1 \times {}^5C_3$

$\vdots$

$d = 98$ sum = $98 \times {}^nC_1 \times {}^{97}C_3$

Total = $4 \cdot {}^{95}C_1 \cdot {}^3C_3 + 5 \cdot {}^{94}C_1 \cdot {}^4C_3 \ldots\ldots \times 98 \cdot {}^1C_1 \cdot {}^{97}C_3$

$= 4\sum_{r=1}^{95} \dfrac{(99-r)}{4} \cdot r\, {}^{98-r}C_3$

$= 4\sum_{r=1}^{95} {}^{99-r}C_4 \times r$

$= 4 \times {}^{100}C_6$

$\therefore$ A.M = $\dfrac{4 \times {}^{100}C_6}{{}^{99}C_5}$

$= \dfrac{200}{3}$

$\therefore \left[\dfrac{200}{3}\right] = 66$

18. (71) Total number of non-intersecting diagonals

$A_1A_3, A_1A_4, A_1A_5, \ldots, A_1A_4 \to 47$

Total number of intersecting diagonals at only one point to the non-intersecting diagonals

$A_2A_4, A_4A_6, A_6A_8, \ldots, A_{48}A_{50} \to 24$

Total = 47 + 24 = 71

19. (92) $A = \{n \in N : p(n) \neq 0, p(n)$ is square free and $s(n)$ is proper divisor of $p(n)\}$

$p(n)$ is square free so number n can containing digit 1, 2, 3, 5, 7 or 1, 5, 7, 6.

$s(n)$ is proper divisor of $p(n)$.

So, max. possible value of $s(n) = 3 \times 5 \times 7 = 105$

For making digit sum 105, n contain digit 2, 3, 5 and 7 one time and digit 1, 88 times.

$s(n) = 2 + 3 + 5 + 7 + 1 \times 88 = 105$

$p(n) = 2 \times 3 \times 5 \times 7 \times 1 \ldots. 1 = 210$

Max. number of digits in $n = 88 + 4 = 92$

20. (43) $A = \{a_1, a_2, a_3, \ldots, a_p\}$

$B = \{b_1, b_2, b_3, \ldots, b_q\}$

$a_{p.q} = 12$

$p.b_q = 11$

Case-A: $p = 11, b_q = 1$

$A = \{a_1, a_2, a_3 \ldots a_{11}\}, B = \{1\}$

$\Rightarrow a_{11} = 12, q = 1$

$\therefore {}^{11}C_{10}$ = total ways

Case-B: $p = 1, b_q = 11$

(1) $A = \{12\}, B = \{11\} \to$ 1 way

(2) $A = \{6\}, B = \{b_1, 11\} \to {}^{10}C_1$ ways

(3) $A = \{4\}, B = \{b_1, b_2, 11\} \to {}^{10}C_2$ ways

(4) $A = \{3\}, B = \{b_1, b_2, b_3, 11\} \to {}^{10}C_3$ ways

(5) $A = \{2\}, B = \{b_1, b_2, b_3, b_4, b_5, 11\} \to {}^{10}C_5$ ways

(6) $A = \{1\}, B = \{b_1, b_2, \ldots, b_{11}, 11\} \to$ 0 ways

$\therefore$ Total ways = 11 + 1 + 10 + 45 + 120 + 252

= 439

= 100 × 4 + 39

$a + b = 43$

21. (15) $\because$ We need $\sum f(i)$ least we will choose n closest to 2023.

$\therefore$ For $\frac{n(n+1)}{2}+\sum f(i)=2023$

Choose $n = 63$

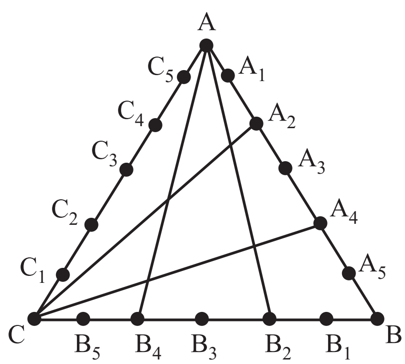
$\Rightarrow$ $2016 + \sum f\text{ (i)} = 2023$

$\Rightarrow$ $\sum f(i) = 7$

Now, all we need is to partition in all possible ways.

No.	No. of partition
7	1
6	1
5	2
4	3
3	4
2	3
1	1
	15

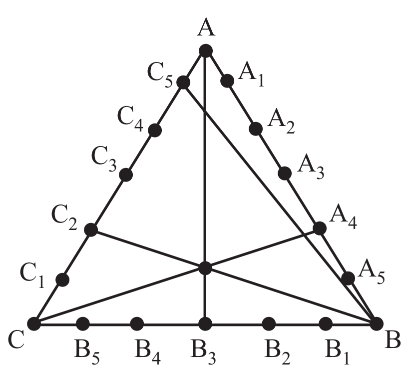
22. (77) **Case I:**



To divide the triangle into 9 regions. Two pegs must be selected from a side and the other two from a different side.

This can be done in ${}^3C_2\times{}^5C_2\times{}^5C_2$ = 300 ways

Case II:



Now, we are choosing 3 points on three sides, such that three lines from those points are concurrent.

By using Ceva's theorem in which product of three different ratio leads to 1.

Possible ratio on side AB, BC and CA will be of the form $\frac{m}{n},\frac{n}{m}$ and 1.

i.e. $\frac{m}{n}\times\frac{n}{m}\times 1=1$

Ratio 1 : 1 can be choosen in 3 ways for all three sides other ratio can be choosen in 4 ways for other two sides i.e. there are $3 \times 4 + 1 = 13$ ways

Fourth point can be choosen in ${}^{12}C_1$, ways

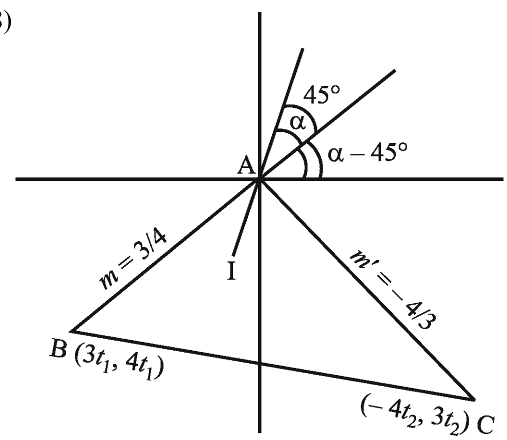
Total such possibilities = $12 \times 13 = 156$ ways

Total ways = 300 + 156 = 456

Sum of squares of digit = $4^2 + 5^2 + 6^2$

= 16 + 25 + 36 = 77

23. (18)



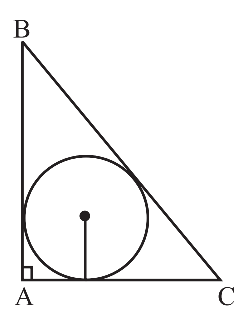
Use concept of shifting coordinate,

Slope of AI = $\frac{84}{12}=7$

Slope of BC = $\tan(\alpha-45^\circ)=\frac{7-1}{1+7}=\frac{3}{4}$

Radius of incircle is $\frac{AI}{\sqrt{2}}=\frac{\sqrt{12^2+84^2}}{\sqrt{2}}=60$

$r = 60$



Use concept,

$$\frac{AB + AC - BC}{2} = r$$

$(BC)^2 = (AC)^2 + (AB)^2$

$(BC)^2 = (AC + AB - 2r)^2$

$(5t_1 + 5t_2 - 2 \times 5 \times 12)^2 = 25\ (t_1^2 + t_2^2)$

$(t_1 + t_2 - 2 \times 12)^2 = t_1^2 + t_2^2$

Put $t_1 = x + 12$

$t_2 = y + 12$

Equation become

$(x - 12)\ (y - 12) = 2 \times 12^2$

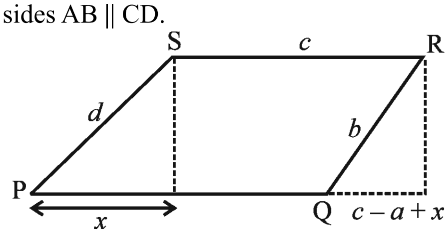
$\geq 0 \qquad \geq 0$

Total number as triangle is equal to pair of (x, y)

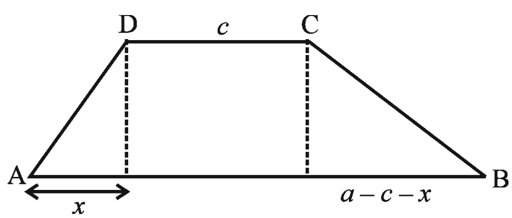
$(x - 12)\ (y - 12) = 2^5 \times 3^2$

No. of pair $(x, y) = 6 \times 3 = 18$

24. (31) Without losing generality, assume $a > c$ and $d > b$ and sides AB || CD.



or



$d^2 - x^2 = b^2 - (c - a + x)^2$,

Similarly, $d^2 - x^2 = b^2 - (a - c - x)^2$,

$$\Rightarrow\ x = \frac{(a-c)^2 + d^2 - b^2}{2(a-c)}$$

$$\Rightarrow\ x = \frac{(a-c)^2 + d^2 - b^2}{2(a-c)}$$

If $x \in (0, d)$, then there will be unique trapezoid.

$$\Rightarrow\ \frac{(a-c)^2 + d^2 - b^2}{2c(c-a)} \in (0,\ d)$$

$\Rightarrow\ (a - c)^2 + d^2 - b^2 - 2d\ (a - c) < 0$

$\Rightarrow\ (a - c - d)^2 - b^2 < 0$

$\Rightarrow\ (a - c - d - b)\ (a - c - d + b) < 0$

$\Rightarrow\ (a - c - d + b) > 0$

$\Rightarrow\ a + b > c + d$

And $a > c$, $d > b$

Using these inequality, numerate these pairs (a, b, c, d)

Case I : $a = 10 \Rightarrow$ Total no. of cases = 16

Case II : $a = 9 \Rightarrow$ Total no. of cases = 9

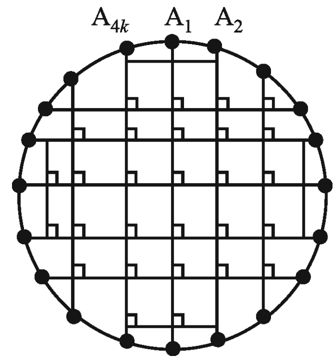
Case III : $a = 8 \Rightarrow$ Total no. of cases = 4

Case IV : $a = 7 \Rightarrow$ (7, 9, 5, 10) and (7, 8, 5, 9)

$\Rightarrow$ 2 cases

$\Rightarrow$ Total = 31

25. (30)



Case-I	**Case-II**
Let $n = 4k$	$n = 4k + 2$
$(1 + 3 + 5 + ...(2k - 1) + ... 3+1).k$	$(1+3+...+2k-1)\times 2\times(2k+1)$
$= (k^2 + (k-1)^2)\ k \geq 1000$	$\Rightarrow (2k+1)\cdot 2k^2 \geq 1000$
$\Rightarrow k\ (2k^2 - 2k + 1) \geq 1000$	$\Rightarrow k^2(2k+1) \geq 500$
$\Rightarrow k \geq 9$ as $k \in N$	$\Rightarrow k \geq 7$
$\Rightarrow 4k \geq 36$	$\Rightarrow n \geq 30$
$\Rightarrow n \geq 36$	$\therefore$ min (36, 30) = 30

26. (19) No. of ways to make 100 Bens. as per Binary land government rules are as follows:

$2^6, 2^5, 2^2$	$2^6, 2^4, 2^3, 2^2, 2^2, 2^1, 2^0, 2^0$
$2^6, 2^5, 2^1, 2^1$	$2^5, 2^5, 2^4, 2^4, 2^2$
$2^6, 2^5, 2^1, 2^0, 2^0$	$2^5, 2^5, 2^4, 2^4, 2^1, 2^1$
$2^6, 2^4, 2^4, 2^2$	$2^5, 2^5, 2^4, 2^4, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^4, 2^1, 2^1$	$2^5, 2^5, 2^4, 2^3, 2^3, 2^2$
$2^6, 2^4, 2^4, 2^1, 2^0, 2^0$	$2^5, 2^5, 2^4, 2^3, 2^3, 2^1, 2^1$
$2^6, 2^4, 2^3, 2^3, 2^2$	$2^5, 2^5, 2^4, 2^3, 2^3, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^3, 2^3, 2^1, 2^1$	$2^5, 2^5, 2^4, 2^3, 2^2, 2^2, 2^1, 2^1$
$2^6, 2^4, 2^3, 2^3, 2^1, 2^0, 2^0$	$2^5, 2^5, 2^4, 2^3, 2^2, 2^2, 2^1, 2^0, 2^0$
$2^6, 2^4, 2^3, 2^2, 2^2, 2^1, 2^1$	

Total 19 possible ways.

27. (91)

$a + c = b + d$	$a < b < d < c$
$a + c = 37$	(20, 17), (18, 19) $\rightarrow 2 \rightarrow 1$ way
$a + c = 36$	(20, 16), (19, 17) $\rightarrow 2 \rightarrow 1$ way
$a + c = 35$	(20, 15), (19, 16), (18, 17) $\rightarrow 3 \rightarrow {}^3C_2$ way

$a + c = 34$ (20, 14), (19, 15), (18, 16) $\to 3$
$\to {}^3C_2$ way

$a + c = 33$ (20, 13), (19, 14) ... (18, 15), (17, 16)
$\to 4 \to {}^4C_2$ way

$a + c = 32$ (20, 12), (19, 13) ... (17, 15) $\to 4$

$a + c = 31$ (20, 11), (16, 15) $\to 5$

$a + c = 30$ (20, 10), (16, 14) $\to 5$

$a + c = 29 \to 6$ (20, 9) ... (15, 14) $\to 6$

$a + c = 28 \to 6$ (20, 8) ... (15, 13) $\to 6$

$a + c = 27 \to 7$ (20, 7) ... (14, 13) $\to 7$

$a + c = 26 \to 7$

$a + c = 25 \to 8$

$a + c = 24 \to 8$

$a + c = 23 \to 9$

$a + c = 22 \to 9$

$a + c = 21 \to 10$ (20, 1) ... (11, 10)

$a + c = 20 \to 9$ (19, 1), (18, 12) ... (11, 9)

$a + c = 19 \to 9$ (18, 1) ... (11, 9)

$a + c = 18 \to 8$ (17, 1), (10, 8)

⋮

⋮

$a + c = 5 \to 2$ (4, 1) (3, 2)

Total balanced quadruple

$= 4\,({}^2C_2 + {}^3C_2 \ldots {}^9C_2) + {}^{10}C_2$

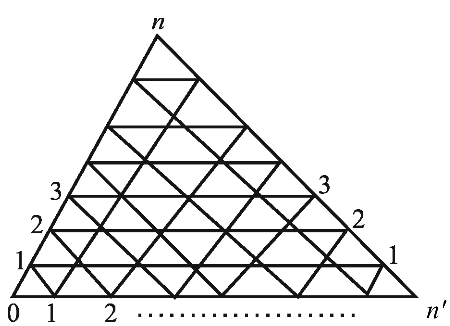
$= 4\,({}^{10}C_3) + {}^{10}C_2$

$= \dfrac{4 \times 10 \times 9 \times 8}{6} + 45 = 525$

${}^{20}C_4 - 4411 = 434$

For least balanced = 525 – 434 = 91

28. (67)



Total number of vertices

$= \dfrac{(n+1)(n+2)}{2}$

Let head be donate by '0' and tail by '1'.

$\therefore$ Initial sum of all vertices = 0.

After each move three '0' becomes three '1'.

Let after k^{th} move the sum be $f(k)$.

$\therefore \quad f(k+1) = f(k) + 3$

Finally, number of tails sum $= \dfrac{(n+1)(n+2)}{2}$

But initially it was zero.

Hence, 3 divides $\dfrac{(n+1)(n+2)}{2}$.

$\therefore \quad n = 3k + 1 \quad$ or $\quad 3k + 2$.

Number of such n which are less than 100 are 67.

29. (95) $99 = 9 \times 11 \times 1 \times 1 \times 1 \times 1 \ldots\ldots \times 1$ (79 times '1')

$= 9 + 11 + 1 + 1 + \ldots\ldots$ 79 times

$= 33 \times 3 \times 1 \times 1 \times 1 \ldots\ldots$ (66 times 1)

$= 33 + 3 + 1 + 1 + \ldots\ldots$ 66 times

Hence, 99 is not beautiful.

$98 = 49 \times 2 \times 1 \times 1 \times \ldots\ldots$ 47 times

$= 49 + 2 + 1 + 1 + \ldots\ldots$ 47 times

$= 14 \times 7 \times 1 \times 1 \times \ldots\ldots$ 77 times

$= 14 + 7 + 1 + 1 + \ldots\ldots$ 77 times

Hence, 98 is not beautiful.

97 is not beautiful as it can not be written in these expanded forms.

$96 = 2 \times 48 \times 1 \times 1 \times \ldots\ldots$ 46 times

$= 2 + 48 + 1 + 1 + \ldots\ldots$ 46 times

$= 3 \times 32 \times 1 \times 1 \times \ldots\ldots$ 61 times

$= 3 + 32 + 1 + 1 + \ldots\ldots$ 61 times

Hence, 96 is not beautiful.

$95 = 19 \times 5 \times 1 \times 1 \times \ldots\ldots$ 71 times

$= 19 + 5 + 1 + 1 + \ldots\ldots$ 71 times

As 95 is uniquely represents, hence it is beautiful.

30. (18) For a number to have odd divisors it must be a perfect square $n \le 2023$. Nearest square is 44^2.

But as this is an even square.

$$\sum_{i-1}^{n=44^2} d(i) \text{ is} \to \text{even}$$

Adding odd number even times makes it even.

$$\therefore \quad \sum_{i=1}^{43^2} d(i) \text{ is odd.}$$

It will remain true for the numbers between

$(44^2 - 43^2) + (42^2 - 41^2) + \ldots. + (9^2 - 1^2)$

$44 + 43 + 42 + 41 + \ldots\ldots + 9 + 1$

$\dfrac{44}{2} \times 45 = 990$

Sum of its digits = 18.