RMO (2023-2024)

(Non-KV & Non-JNV)

Time: 3 hours

INSTRUCTIONS

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let $\mathbb{N}$ be the set of all natural numbers and $S = \{(a, b, c, d) \in \mathbb{N}^4 : a^2 + b^2 + c^2 = d^2\}$. Find the largest positive integer m such that m divides abcd for all $(a, b, c, d) \in S$.

2. Let ω be a semicircle with AB as the bounding diameter and let CD be a variable chord of the semicircle of constant length such that C, D lie in the interior of the arc AB. Let E be a point on the diameter AB such that CE and DE are equally inclined to the line AB. Prove that
   (a) the measure of $\angle CED$ is a constant;
   (b) the circumcircle of triangle CED passes through a fixed point.

3. For any natural number n, expressed in base 10, let s(n) denote the sum of all its digits. Find all natural numbers m and n such that m < n and $(s(n))^2 = m$ and $(s(m))^2 = n$.

4. Let Ω_1, Ω_2 be two intersecting circles with centres O_1, O_2 respectively. Let l be a line that intersects Ω_1 at points A, C and Ω_2 at points B, D such that A, B, C, D are collinear in that order. Let the perpendicular bisector of segment AB intersect Ω_1 at points P, Q; and the perpendicular bisector of segment CD intersect Ω_2 at points R, S such that P, R are on the same side of l. Prove that the midpoints of PR, QS and O_1O_2 are collinear.

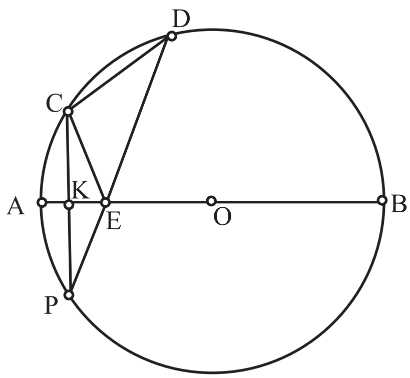
5. Let n > k > 1 be positive integers. Determine all positive real numbers $a_1, a_2......, a_n$ which satisfy

$$\sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}}=\sum_{i=1}^{n}a_i = n.$$

6. Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

Hints & Solutions

1. Since $d^2 \equiv 0, 1 \pmod 4$ at most one of a, b, c is odd. Therefore 4 divides abcd. Also, if 3 does not divide each of a, b and c then
$d^2 = a^2 + b^2 + c^2 \equiv 1 + 1 + 1 \quad 0 \pmod 3$.
Thus 3 divides abcd. Therefore 12 divides abcd and if m is the largest positive integer such that m divides all abcd $\in$ S then m = 12k for some positive integer k. But (1, 2, 2, 3) $\in$ S and 1, 2, 2, 3 = 12. Hence k = 1 and m = 12.

2. 

Construct the circle with AB as diameter and let this circle be Ω. Draw CK $\perp$ AB with K on AB. Let CK produced meet Ω again in P. Join EP. Observe that

$\angle DEB = \angle CEK = \angle PEK$.

Hence $\angle PEK + \angle CEK + \angle CED = 180$.

Therefore P, E, D are collinear. This shows that

$$\angle CED = 2\angle CPD$$

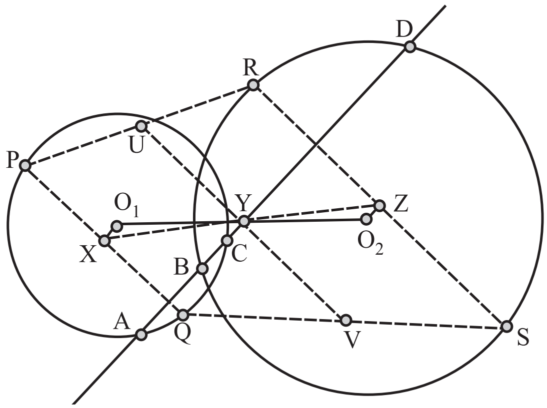
is a constant. If O is the centre of Ω then we get $\angle COD = 2\angle CPD = \angle CED$. Hence the circumcircle of triangle CED passes through O which is a fixed point.

3. Let m < n be such natural numbers. Let
$m = 10^{k-1}a_{k-1} + 10^{k-2}a_{k-2} + \cdots + 10a_1 + a_0$
be a k-digit number. Then we have
$10^{k-1} \le m < n = s(m)^2$
$= (a_{k-1} + a_{k-2} + \ldots + a_1 + a_0)^2 \le 9^2k^2$:
If $k \ge 5$, this is not possible. Hence $k \le 4$.
If k = 4, then
$m = 1000a_3 + 100a_2 + 10a_1 + a_0 < (a_3 + a_2 + a_1 + a_0)^2 \le 36^2 = 1296$:
This shows that $a_3 = 1$. In this case
$m = 1000 + 100a_2 + 10a_1 + a_0 < (1 + a_2 + a_1 + a_0)^2 \le 28^2 = 784$;
which is impossible. Hence m must be a 3-digit number. Again
$m = 100a_2 + 10a_1 + a_0 < (a_2 + a_1 + a_0)^2 \le 27^2 = 729$.
Hence $a_2 \le 7$. If $a_2 = 7$, then
$m = 700 + 10a_1 + a_0 < (7 + a_1 + a_0)^2 \le 25^2 = 625$,
which is not possible. Similarly, $a_2 = 6$ gives
$m = 600 + 10a_1 + a_0 < (6 + a_1 + a_0)^2 \le 24^2 = 576$;
which again is impossible. If $a_2 = 5$, we obtain the maximal digital sum 23 when $a_1 = a_0 = 9$. Otherwise $s(m) \le 22$ and
$m = 500 + 10a_1 + a_0 < n = s(m)^2 \le 22^2 = 484$.
Thus we can also rule out $a_2 = 5$. Therefore $a_2 \le 4$. This means, $m \le 22^2$.
Now we can search which squares up to 22^2 admit an n such that $m = (s(n))^2$ and $n = (s(m))^2$. The first such square is $m = 81 = 9^2$. But in this case $n = s(m)^2 = 81$. But now m = n violating m < n.

The next square is m = 169 = 13^2. In this case $s(m)^2 = 16^2 = 256 = n$ and $s(n)^2 = 13^2 = 169$.

Thereafter, no square satisfies this. Thus we get the pair (m, n) = (169, 256).

4.



Let the midpoints of segments PQ, O_1O_2, RS be denoted by X, Y, Z respectively.

We observe that B is the reflection of A in line PQ. Hence B is the orthocentre of ΔCPQ.

Hence, $O_1X = BC/2$. Similarly, $O_2Z = BC/2$.

By the S-A-S test, $\Delta XO_1Y \cong \Delta ZO_2Y$; hence X – Y – Z with XY = Y Z.

The endpoints of segments PR, XZ, QS lie on parallel lines PQ and RS, so their midpoints are collinear.

5. By A.M – G.M inequality we have

$$\frac{(k-1)a_i^k+1}{k} \ge \left(a_i^{k(k-1)}\right)^{1/k} = a_i^{k-1}$$

which implies $\sqrt{\dfrac{ka_i^k+1}{(k-1)a_i^k+1}} \le \sqrt{a_i}$. Hence

$$\sum_{i=1}^{n}\sqrt{a_i} \ge \sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_i = n.$$

But by Cauchy-Schwarz inequality we have

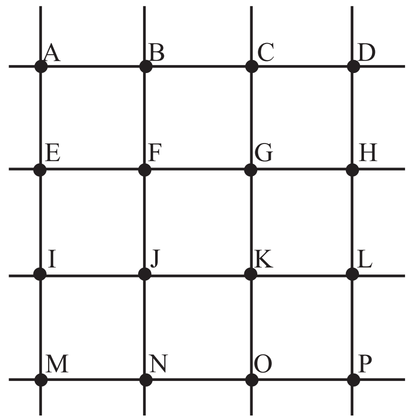
$$\sum_{i=1}^{n}\sqrt{a_i} \le \sqrt{n\left(\sum_{i=1}^{n} a_i\right)} = n.$$

Therefore

$$n \ge \sum_{i=1}^{n}\sqrt{a_i} \ge \sum_{i=1}^{n}\sqrt{\frac{ka_i^k}{(k-1)a_i^k+1}} = \sum_{i=1}^{n} a_i = n$$

and hence equality holds everywhere which implies $a_i = 1$ for i = 1, 2, ..., n.

6.



Let us label the points as illustrated in the above diagram. We can consider the following cases:

Case 1: None of the central 4 points {F, G, J,K} is colored.

We can partition the remaining 12 points into the 3 sets {A, D, P, M}, {B, H, O, I}, {C, L, N, E}.

By PHP, at least 3 of the 7 colored points lie in the same set; forming a 45 – 45 – 90 triangle (an isosceles right-angled triangle).

Case 2: At least one of the central 4 points is colored; WLOG let point F be colored.

Subcase 2.1: Points F, C are both colored.

Then, none of the points A, B, G, H, K can be colored, as each of them forms a 45–45–90 triangle along with F, C. The remaining 9 points (out of which 5 are colored) can be partitioned into the 4 sets {E, I, J}, {D, O}, {L, M}, {N, P}. So by PHP, some set contains atleast 2 colored points, which form a 45 – 45 – 90 triangle along with F.

Subcase 2.2: Point F, I are both colored. By symmetry, this is identical to subcase 2.1.

Subcase 2.3: Point F is colored, but neither C nor I is colored.

Then apart from C, F, I, the remaining 13 points (out of which 6 are colored) can be partitioned into the 5 sets {A, B, E}, {G, J, K}, {D, O}, {H, N, P}, {L, M}. So by PHP, some set contains atleast 2 colored points, which form a 45 – 45 – 90 triangle along with F.