IOQM (2024-25)

Time: 3 hours **Max. Marks: 100**

INSTRUCTIONS

1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is STRICTLY PROHIBITED. Only ordinary pens and pencils are allowed inside the examination hall.

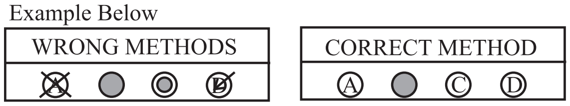
2. The correction is done by machines through scanning. On the OMR sheet, darken bubbles completely with a **black** or **blue ball pen**. Please DO NOT use a pencil or a gel pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

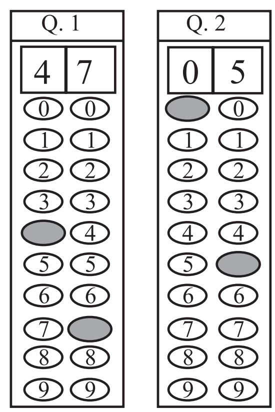
4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS
1. "Think before your ink"
2. Marking should be done with Blue/ Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below



4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. The smallest positive integer that does not divide $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$ is

2. The number of four-digit odd numbers having digits 1, 2, 3, 4, each occurring exactly once, is

3. The number obtained by taking the last two digits of 5^{2024} in the same order is

4. Let ABCD be quadrilateral with $\angle ADC = 70°$, $\angle ACD = 70°$, $\angle ACB = 10°$ and $\angle BAD = 110°$. The measure of $\angle CAB$ (in degrees) is

5. Let $a = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, let $b = \frac{x}{z}+\frac{y}{x}+\frac{z}{y}$ and let $c = \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)$. The value of $|ab - c|$ is

6. Find the number of triples of real numbers (a, b, c) such that $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$

7. Determine the sum of all possible surface areas of a cube two of whose vertices are (1, 2, 0) and (3, 3, 2).

8. Let n, be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of (n + 1) is divisible by 5. What are the first two digits of n in the same order?

9. Consider the grid of points $X = \{(m, n) \mid 0 \le m, n \le 4\}$. We say a pair of points $\{(a, b), (c, d)\}$ in X is a knight move pair if ($c = a \pm 2$ and $d = b \pm 1$) or ($c = a \pm 1$ and $d = b \pm 2$). The number of knight-move pairs in X is

10. Determine the number of positive integral value of p for which there exists a triangle with side a, b and c which satisfy $a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0$

11. The positive real numbers a, b, c satisfy:

$$\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1$$

$$\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2$$

What is the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$?

12. Consider a square ABCD of side length 16. Let E, F be points on CD such that CE = EF = FD. Let the line BF and AE meet in M. The area of ΔMAB is

13. Three positive integers a, b, c with a > c satisfy the following equations:

$ac + b + c = bc + a + 66$, $a + b + c = 32$.

Find the value of a.

14. Initially, there are 3^{80} particles at the origin (0, 0). At each step the particles are moved to points above the x-axis as follows: if there are n particles at any point (x, y), then $\left[\frac{n}{3}\right]$ of them are moved to (x + 1, y + 1), $\left[\frac{n}{3}\right]$ are moved to (x, y + 1) and the remaining to (x – 1, y + 1). For example, after the first step, there are 3^{79} particles each at (1, 1), (0, 1) and (–1, 1). After the second step, there are 3^{78} particles each at (–2, 2) and (2, 2), 2×3^{78} particles each at (–1, 2) and (1, 2), and 3^{79} particle at (0, 2). After 80 steps, the number of particles at (79, 80) is:

15. Let X be the set consisting of twenty positive integers n, n + 2,…., n + 38. The smallest value of n for which any three numbers a, b, c, ∈ X, not necessarily distinct, form the sides of an acute-angled triangle is:

16. Let $f : R \to R$ be a function satisfying the relation $4f(3 - x) + 3f(x) = x^2$ for any real x. Find the value of f(27) – f(25) to the nearest integer. (Here R denotes the set of real numbers.)

17. Consider an isosceles triangle ABC with sides BC = 30, CA = AB = 20. Let D be the foot of the perpendicular from A to BC, and let M be the midpoint of AD. Let PQ be a chord of the circumcircle of triangle ABC, such that M lies on PQ is parallel to BC. The length of PQ is:

18. Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if gcd(p, q) = 1 and p + q divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in the same order).

19. Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

20. On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example starting with 8 you can reach 13 as follows: $8 \to 16 \to 13$. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

21. An integer n is such that $\left[\frac{n}{9}\right]$ is a three-digit number with equal digits, and $\left[\frac{n-172}{4}\right]$ is a 4 digit number with the digits 2, 0, 2, 4 in some order. What is the remainder when n is divided by 100 ?

22. In a triangle ABC, $\angle BAC = 90°$. Let D be the point on BC such that AB + BD = AC + CD. Suppose BD : DC = 2 : 1. If $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$, where m, n are relatively prime positive integers and p is a prime number, determine the value of m + n + p.

23. Consider the fourteen numbers, 1^4, 2^4 ,….,14^4. The smallest natural number n such that they leave distinct remainders when divided by n is:

24. Consider the set F of all polynomials whose coefficients are in the set of {0, 1}. Let $q(x) = x^3 + x + 1$. The number of polynomials p(x) in F of degree 14 such that the product p(x)q(x) is also in F is:

25. A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N?

26. The sum of [x] for all real numbers x satisfying the equation $16 + 15x + 15x^2 = [x]^3$ is:

27. In a triangle ABC, a point P in the interior of ABC is such that $\angle BPC - \angle BAC = \angle CPA - \angle CBA = \angle APB - \angle ACB$. Suppose $\angle BAC = 30°$ and AP = 12. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If $m\sqrt{n}$ in the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn?

28. Find the largest positive integer n < 30 such that $\frac{1}{2}(n^8 + 3n^4 - 4)$ is not divisible by the square of any prime number.

29. Let $n = 2^{19} 3^{12}$. Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

30. Let ABC be a right-angled triangle with $\angle B = 90°$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeter of triangle ABC are integers?

Answers

1. (11)	2. (12)	3. (25)	4. (70)	5. (01)	6. (06)	7. (99)	8. (49)	9. (48)	10. (05)
11. (12)	12.(96)	13. (19)	14.(80)	15. (92)	16. (08)	17. (25)	18. (13)	19. (12)	20. (10)
21. (91)	22.(34)	23. (31)	24.(50)	25. (22)	26. (33)	27. (27)	28. (20)	29. (28)	30. (25)

Hints & Solutions

1. **(11)** n! is divisible by all prime numbers less than or equal to n. Thus 11 is the smallest prime number that does not divide 9!

2. **(12)** abcd

$\therefore$ abcd is odd number therefore d can take values as 1 or 3 $\Rightarrow$ 2 possibility and

a can take 3 values

b can take 2 values

c can take 1 value

Total values = 2 × 3 × 2 × 1

$\Rightarrow$ 12 numbers

3. **(25)** Last two digits of 5^{2024}

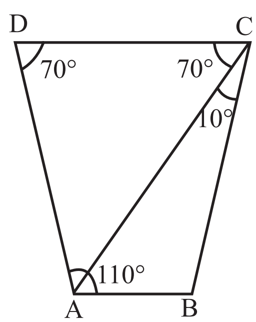
$\Rightarrow$ $5^{2024} \equiv a \pmod{100}$

$5^4 \equiv 25 \pmod{100}$

$5^{2024} \equiv 25 \pmod{100}$

$\Rightarrow$ last 2 digits in same order is 25

4. **(70)** In ΔACD



$\angle CAD = 180° - \angle ACD - \angle ADC = 40°$

$\angle CAB = \angle BAD - \angle CAD$

$= 110° - 40° = 70°$

5. **(01)** $a = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$ ...(i)

and $b = \frac{x}{z} + \frac{y}{x} + \frac{z}{y}$ ... (ii)

and $c = \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$

$= \left(a - \frac{z}{x}\right)\left(a - \frac{x}{y}\right)\left(a - \frac{y}{z}\right)$

$= a^3 - \left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right)a^2 + \left(\frac{z}{y} + \frac{x}{z} + \frac{y}{x}\right)a - 1$

$= a^3 - a^3 + ab - 1$

$\therefore$ $1 = ab - c$

6. **(06)** $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$

If both equation have to satisfy

$\Rightarrow$ Either a or b or c = ± 1

And other have to be 0.

So, if a = 1

$\Rightarrow$ b = c = 0

If a = – 1

$\Rightarrow$ b = c = 0

Similarly, for b and c if b = ± 1.

$\Rightarrow$ a = c = 0

And if c = ± 1

$\Rightarrow$ a = b = 0

Total triples = 6.

7. **(99)** Distance between (1, 2, 0) and (3, 3, 2) is

$\sqrt{2^2 + 1^2 + 2^2} = 3$

Now, line joining the vertices of a cube are the following : (a is edge length)

(i) edge = a

(ii) face - diagonal $= \sqrt{2}a$

(iii) body - diagonal $= \sqrt{3}a$

$\Rightarrow$ possible values of a are

$a = 3, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{3}}$

$\Rightarrow$ possible values of surface area are

S. A. = $6a^2$ = 54, 27, 18

Sum = 99

8. **(49)** Let sum of digits of n number = 5λ

Let sum of digits of n + 1 number = 5k

$\underbrace{1, 2}_{1}, \underbrace{3}_{1}, ..., \underbrace{9, 10}_{8}, \underbrace{11}_{1}, ..., \underbrace{19, 20}_{8}, ..., \underbrace{99, 100}_{17}$

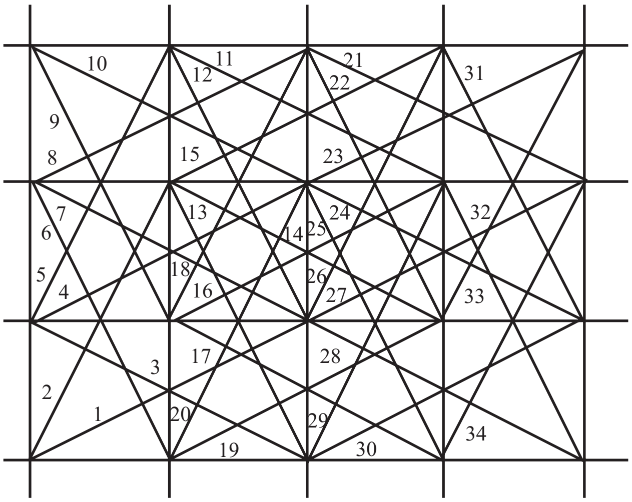
$...., \underbrace{999, 1000}_{26}, ..., \underbrace{9999, 10,000}_{35}$

$..., \underbrace{49999, 50000}_{\text{diff } 35}$

$\therefore$ Smallest number = 49,999

First two digits = 49.

9. **(48)**



Consider the grid as shown.

Each line represents the allowed move from one and point of line to other end point of line we can see the movement allowed in 3 × 2, 2 × 2 rectangle, vertical or horizontal. In each rectangle there are 4 moves allowed and total number of rectangle are 12.

$\therefore$ Total allowed moved = 12 × 4 = 48.

10. **(05)** $a^2 + p^2b^2 + 9b^2 + 9c^2 - 6ab - 6pbc = 0$

$(a^2 + 9b^2 - 6ab) + p^2b^2 + 9c^2 - 6pbc = 0$

$(a - 3b)^2 + (pb - 3c)^2 = 0$

$a = 3b$; $pb = 3c$

$\Rightarrow \frac{a}{9} = \frac{b}{3} = \frac{c}{p}$

$\Rightarrow |9 - 3| < p < 9 + 3$

So, $6 < p < 12$

Positive integral possible values of $p = \{7, 8, 9, 10, 12\}$

$\therefore$ Number of positive integral values of p is 5.

11. **(12)** $\frac{3c}{a+1} + \frac{a}{2b+1} + \frac{2b}{3c+1} = 1$ ...(i)

$\frac{1}{a+1} + \frac{1}{2b+1} + \frac{1}{3c+1} = 2$ ...(ii)

Add (i) and (ii)

$\Rightarrow \frac{3c+1}{a+1} + \frac{a+1}{2b+1} + \frac{2b+1}{3c+1} = 3$

$\Rightarrow$ A.M. ≥ G. M.

$$\Rightarrow \left(\frac{\frac{3c+1}{a+1} + \frac{a+1}{2b+1} + \frac{2b+1}{3c+1}}{3}\right) \geq (1)^{\frac{1}{3}}$$

$\Rightarrow \frac{3c+1}{a+1} = 1$ and $\frac{2b+1}{3c+1} = 1$

$\frac{a+1}{2b+1} = 1$

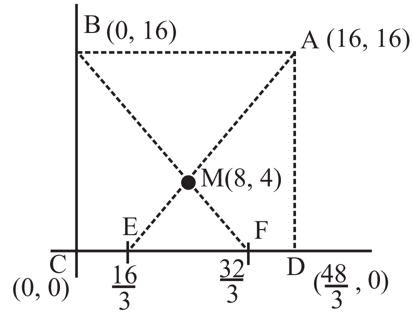
$\Rightarrow (a + 1) = (2b + 1) = (3c + 1)) = k$

$\Rightarrow \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = 2 \Rightarrow \frac{3}{k} = 2 \Rightarrow k = \frac{3}{2}$

$\Rightarrow a = \frac{1}{2}, b = \frac{1}{4}, c = \frac{1}{6}$

$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2 + 4 + 6 = 12$

12. **(96)** Eq of line AE : $2y - 3x + 16 = 0$



Eq of line BF : $2y + 3x - 32 = 0$

$\Rightarrow M \equiv (8, 4)$

Area of triangle AMB

$$\left|\frac{1}{2}\begin{vmatrix} 0 & 16 & 1 \\ 16 & 16 & 1 \\ 8 & 4 & 1 \end{vmatrix}\right| = \frac{1}{2}\left(-16(8) + 1(64 - 8 \times 16)\right)$$

$$\left| = \frac{1}{2}(-16 \times 16 + 64)\right|$$

$$= \left|\frac{16}{2}(4 - 16)\right| = 8 \times 12$$

= 96 sq. units

13. **(19)** $a > c$

$ac + b + c = bc + a + 66$, $a + b + c = 32$

$ac + b + c - bc - a = 66$

$a(c - 1)) - b(c - 1) + 1 (c - 1) = 65$

$(c - 1) (a - b + 1) = 65$

$c = 6$; $a - b = 12$; $a + b = 26$, $a = 19$, $b = 7$

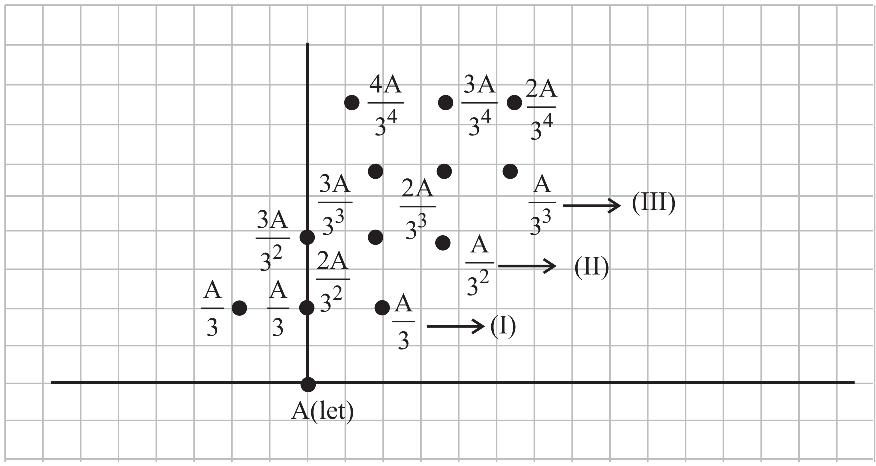
$c = 14$; $a - b = 4$; $a + b = 18$, $a = 11$, $b = 7$

$c = 2$; $a - b = 64$; $a + b = 30$, $a = 47$, $b = -17$ [Not possible]

$c = 66$; $a - b = 0$; $a + b = -34$ [Not possible]

$\because$ $a > c$ so only possibility is $a = 19$

14. (80)



At n^{th} step particle at (n – 1, n) is $\frac{nA}{3^n}$

$\therefore$ At 80^{th} step particles at (79, 80) is $\frac{80A}{3^{80}} = 80$

(where $A = 3^{80}$)

$\therefore$ 80

15. (92) $X = \{n, n + 2, ..., n + 38\}$

a, b, c, $\in$ X

For any a, b, c

(i) Triangle should be formed

(ii) Triangle should be acute

$\rightarrow$ only one angle can be obtuse at max

(i) let $a \le b \le c$

$\Rightarrow$ for triangle

a + b > c for all possible combination

$\Rightarrow$ even if a, b are smallest a = b = n

$\Rightarrow$ n + n > n + 38

$\Rightarrow$ n > 38 $\Rightarrow$ triangle will form

(ii) now using cosine formula largest side longest angle

$\Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab} > 0$ for acute Δ

$\Rightarrow a^2 + b^2 - c^2 > 0$ for acute Δ, $\forall$ a, b, c $\in$ x

$n^2 + (n)^2 - (n + 38)^2 > 0$

$\Rightarrow n^2 - 76n - 38^2 > 0$

$\Rightarrow$ n > 91. 74

$\Rightarrow$ n = 92

16. (08) $4f(3 - x) + 3f(x) = x^2, \forall x \in R$

$4f(3 - (3 - x)) + 3f(3 - x) = (3 - x)^2$

$= 4f(x) + 3f(3 - x) = (x - 3)^2$

$\Rightarrow 12f(3 - x) + 9f(x) = 3x^2$

$12f(3 - x) + 16f(x) = 4(x - 3)^2$

$\Rightarrow 7f(x) = 4(x - 3)^2 - 3x^2$

$7(f(27) - f(25))$

$= (4(24)^2 - 3.27^2) - (4(22)^2 - 3.25^2)$

$= 4(24^2 - 22^2) + 3(25^2 - 27^2)$

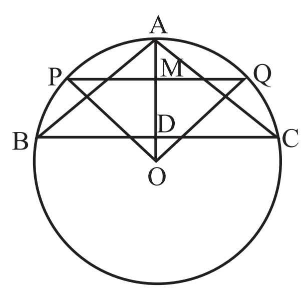
$= 4(46)(2) + 3(52)(-2)$

$= 8 \times 46 - 6 \times 52$

$\Rightarrow f(27) - f(25) = \frac{1}{7}(46 \times 8 - 6 \times 52) = \left(\frac{56}{7}\right) = 8$

17. (25)

$AD = \sqrt{AC^2 - DC^2}$



$= 5\sqrt{7}$

$AM = \frac{AD}{2} = \frac{5}{2}\sqrt{7}$

In ΔABC

a = 30, b = 20

c = 20,

$s = \frac{20 + 20 + 30}{2} = 35$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= 75\sqrt{7}$

Circumradius of ΔABC

$R = \frac{abc}{4\Delta} = \frac{40}{7}\sqrt{7}$

$\Rightarrow$ OA = OP = OQ $= \frac{40}{7}\sqrt{7}$

$\Rightarrow$ OM = R – AM

$= \frac{40}{7}\sqrt{7} - \frac{5}{2}\sqrt{7} = \frac{45}{2\sqrt{7}}$

$PQ = 2QM = 2\sqrt{OQ^2 - OM^2} = 5\sqrt{25} = 25$

18. **(13)**

$r = 100p + q$

$\because$ $p + q \mid r$

$\Rightarrow$ $p + q \mid 100p + q$

$\Rightarrow$ $p + q \mid 99p + p + q$

$\Rightarrow$ $p + q \mid 99p$

But gcd (p + q, p) = 1

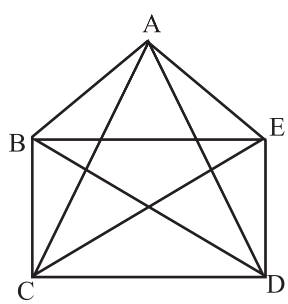
$\Rightarrow$ $p + q \mid 99$

$\therefore$ p + q = 33 or 99

For N to be maximum p + q = 99

Where p = 86, q = 13

19. **(12) Case-I**

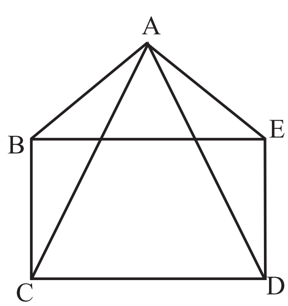


Exactly 5 sides are same color

2 ways (corresponding to each color)

As color of other sides got fixed

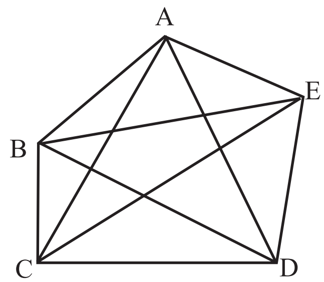
Case-II



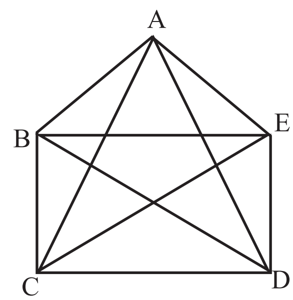
Exactly 4 sides are same color

Not possible $\therefore$ 0 ways as color of other sides got fixed and there will be one triangle which will have all sides same color

Case-III



Exactly 3 consecutive sides are same color



Exactly 3 non consecutive sides are same color

Only one way of colouring as color of other sides got fixed

$\therefore$ $^5C_1 \times 2$ ways (corresponding to each color for shown figure)

$\therefore$ Total number of ways

$= 2 + {}^5C_1 \times 2 = 12$ ways

20. **(10)** $11 - 3 = 8$ (1)

$8 - 3 = 5$ (2)

$5 \times 2 = 10$ (3)

$10 \times 2 = 20$ (4)

$20 - 3 = 17$ (5)

$17 \times 2 = 34$ (6)

$34 - 3 = 31$ (7)

$31 \times 2 = 62$ (8)

$62 \times 2 = 124$ (9)

$124 - 3 = 121$ (10)

It is better to go from 121 to 11 notice that x(+3) (+3) (÷2) = x(÷2) (+3) without BODMAS

a_n is odd it cant be divided by 2 thus 3 must be added adding 3 more than once will just waste steps Thus the following rules will lead to most optimal path from 121 to 11

$$a_{n+1} = \begin{cases} a_n + 3; 2 \mid a_n \\ a_n / 2; 2 \mid a_n \end{cases}$$

121 > 124 > 62 > 31 > 34 > 17 > 20 > 10 > 5 > 8 > 11

21. (91) Let $\left[\frac{n-172}{4}\right] = k, k \in I$

$\Rightarrow \frac{n-172}{4} \in [k, k+1)$

$n - 172 \in [4k, 4k + 4)$

$n \in [4k + 172, 4k + 176)$

$\frac{n}{9} \in \left[\frac{4k+172}{9}, \frac{4k+176}{9}\right)$

Now, $\left[\frac{n}{9}\right] \in \{111, 222, 333, ..., 999\}$

$\Rightarrow \frac{4k+176}{9} > \overline{aaa} > \frac{4k+172}{9}$,

for $a \in \{1, ..., 9\}$

For a = 9

$k \in (2203.75, 2204.75)$

$\Rightarrow k = 2204$

$\left[\frac{n-172}{4}\right] = 2204$ ...(i)

And $\left[\frac{n}{9}\right] = 999$ ...(ii)

From (i)

$\frac{n-172}{4} \in [2204, 2205)$

$\Rightarrow n \in [8988, 8992)$

From (ii)

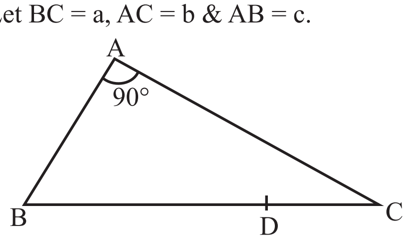
$\frac{n}{9} \in [999, 1000)$

$\Rightarrow n \in [8991, 9000)$

$\Rightarrow n = 8991$

$\Rightarrow 91$

22. (34) Let BC = a, AC = b & AB = c.



AB + BD = AC + CD

$\Rightarrow$ D is point of contact of A-excircle with BC

$\Rightarrow BD = s - c$ & $CD = s - b$

Given $\frac{BD}{DC} = \frac{2}{1} \Rightarrow \frac{s-c}{s-b} = \frac{2}{1} \Rightarrow \frac{a+b-c}{a+c-b} = \frac{2}{1}$

$\Rightarrow a + 3c = 3b$ .... (1)

Also given $a^2 = b^2 + c^2$ ....(2)

$\Rightarrow (3b - 3c)^2 = b^2 + c^2 \Rightarrow 8b^2 + 8c^2 - 18bc = 0$

$\Rightarrow 4\left(\frac{b}{c}\right)^2 - 9\left(\frac{b}{c}\right) + 4 = 0 \Rightarrow \frac{b}{c} = \frac{9 \pm \sqrt{81-64}}{8}$

$\left(\frac{AC}{AB}\right) = \frac{9+\sqrt{17}}{8} = \frac{m+\sqrt{p}}{n}$

$\Rightarrow m + p + n = 9 + 17 + 8 = 34$

23. (31) $1^4, 2^4$ ...., 14^4

$x^4 \equiv a \pmod{n}$

$y^4 \equiv b \pmod{n}$ such that $a \neq b$ for $x \neq y$ and $x, y \in \{1, 2, ... 14\}$

$(x^4 - y^4) \equiv (a - b) \pmod{n}$

$\Rightarrow (x - y)(x + y)(x^2 + y^2) \equiv (a - b) \pmod{n}$

$\Rightarrow n \mid (x - y)(x + y)(x^2 + y^2)$ ...(i)

We have to find minimum n with condition (i)

Clearly, $n > 27$ as $(x + y) \in \{3, ... 27\}$

Now n = 28, x = 6, y = 8 works

n = 29, x = 5, y = 2 works

n = 30, x = 8, y = 2 works

for x = 31, there are no such x, y,

$31 \mid \underbrace{(x-y)(x+y)(x^2+y^2)}_{\text{Must be prime factor}}$

$31 \mid (x^2 + y^2)$ and $31 \mid (x - y)(x + y)$

$\Rightarrow$ 31 will be the answer

24. (50) $p(x)\,q(x) = (x^{14} + ...)(x^3 + x + 1)$

$p(x) = x^{14} \rightarrow$ 1 case

$p(x) = x^{14} + x^{\alpha}$

$\Rightarrow \alpha = 10, 9, 8, ..., 1, 0 \rightarrow$ 11 case

$p(x) = x^{14} + x^{\alpha} + x^{\beta}$

$$\left.\begin{array}{ll} \alpha = 10, & \beta = 6,5,4,3,2,1,0 \\ \alpha = 9 & \beta = 5,4,3,2,1,0 \\ \alpha = 8 & \beta = 4,3,2,1,0 \\ \alpha = 7, & \beta = 3,2,1,0 \\ \alpha = 6, & \beta = 2,1,0 \\ \alpha = 5, & \beta = 1,0 \\ \alpha = 4, & \beta = 0 \end{array}\right\} \text{28 cases}$$

$p(x) = x^{14} + x^{\alpha} + x^{\beta} + x^{r}$

$$\left.\begin{array}{lll} \alpha = 10, & \beta = 6, & r = 2,1,0 \\ \alpha = 10, & \beta = 5, & r = 1,0 \\ \alpha = 10, & \beta = 4, & r = 0 \end{array}\right\} \text{6 cases}$$

$$\left.\begin{array}{ll}\alpha = 9, & \beta = 5, \quad r = 1, 0 \\ & \beta = 4, \quad r = 0\end{array}\right\} 3 \text{ cases}$$

$\alpha = 8, \beta = 4, r = 0\}$ 1 cases

Hence, total cases = 1 + 11 + 28 + 6 + 3 + 1

= 50 cases.

25. **(22)** $M - \{92\} = \{x_1^2, x_2^2, ... x_m^2\}$

Let K is sum of element of set M – {92}

$\frac{K}{m} = 84, \frac{K+92}{m+1} = 85$

$84m + 92 = 85m + 85$

$m = 7$

$K = 588$

Now $x_1^2 + x_2^2 + ... + x_7^2 = 588$

$22^2 + 7^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$

Largest N^2 is 22^2

$\Rightarrow N = 22$

26. **(33)** $[x]^3 = 15x^2 + 15x + 16$

$[x] \in (x - 1, x]$

$[x]^3 \in ((x-1)^3, x^3]$

$\Rightarrow 15x^2 + 15x + 16 \in ((x-1)^3, x^3]$

$x^3 \geq 15x^2 + 15x + 16$

$\Rightarrow x^3 - 15x^2 - 15x - 16 \geq 0$

$(x - 16)(x^2 + x + 1) \geq 0$

$\Rightarrow x \geq 16$

Also, $15x^2 + 15x + 16 > x^3 - 3x^2 + 3x - 1$

$\Rightarrow x^3 - 18x^2 - 12x - 17 < 0$

Let $f(x) = x^3 - 18x^2 - 12x - 17$

$f'(x) = 3x^2 - 36x - 12$

$= 3(x^2 - 12x - 4)$ (roots α, β) $\alpha < \beta$

$f(\alpha) < 0$

$\Rightarrow$ f(x) has only one real root

$\Rightarrow f(18) = -233$

$f(19) = 116$

$\Rightarrow f(\alpha) = 0$

$\Rightarrow \alpha \in (18, 19)$

For this equation

$x \in [16, 19)$

$\Rightarrow [x] = 16, 17, 18$

(i) $[x] = 16$

$\Rightarrow 15x^2 + 15x + 16 = 16^3$

$(x - 16)(x + 17) = 0$

$\Rightarrow x = 16$ satisfies

(ii) $[x] = 17$

$\Rightarrow 15x^2 + 15x + 16 = 17^3$

$\Rightarrow x = \frac{-15 \pm \sqrt{15^2 - 4.16.15}}{30}$

$\Rightarrow [x] = 17$ satisfies

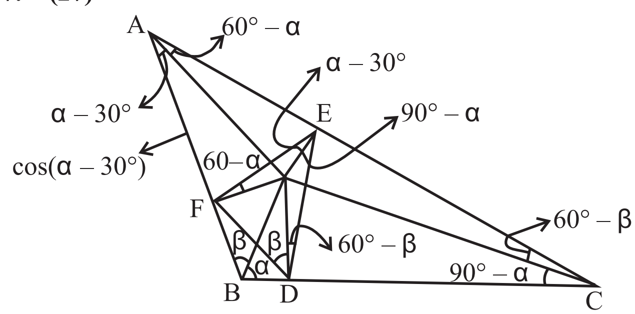
(iii) $[x] = 18$

$\Rightarrow x > 19$ and $x < -20$

$\Rightarrow$ No received value of x

$\Rightarrow$ Sum of $[x] = 16 + 17 = 33$.

27. **(27)**



$EF^2 = I^2 [\cos^2(\alpha - 30°) + \cos^2(60° - \alpha) - \frac{2\sqrt{3}}{2} \cos(\alpha - 30°)\cos(\alpha - 60°)]$

$$= I^2\left[\left(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right)^2 + \left(\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)^2 - \frac{\sqrt{3}}{2}\left(\cos(2\alpha - 90°) + \frac{\sqrt{3}}{2}\right)\right]$$

$$= I^2\left[\cos^2\alpha + \sin^2\alpha + \frac{\sqrt{3}}{2}\sin\alpha\cos\alpha - \frac{3}{4}\right]$$

$$EF^2 = \frac{I^2}{4}, \text{area} = \frac{\sqrt{3}}{4}\cdot EF^2 = \frac{\sqrt{3}}{4}\frac{144}{4} = 9\sqrt{3}$$

28. **(20)** Let $f(n) = \frac{1}{2}(n^8 + 3n^4 - 4)$

$= \frac{1}{2}(n-1)(n+1)(n^2+1)((n^2+1)+1)((n-1)^2+1)$

For $n = 2k + 1$

$\Rightarrow f(2k+1) = \frac{1}{2}(2k)\,2(k+1)(4k^2 + 4k + 2)...$

Clearly $4 | f(2k + 1)$

$\Rightarrow$ Only even cases

$\Rightarrow n = 28, f(28) = \frac{1}{2} \times 27 \times 29 \times (28^2 + 1)(27^2 + 1)(29^2+1)$

Clearly $3^2 | f(28)$

$n = 26, f(26) = \frac{1}{2} \times 25 \times 27 \times (25^2 + 1)(26^2 + 1)(27^2 + 1)$

Clearly $5^2 | f(26)$

$n = 24,\ f(24) = \frac{1}{2}.23 \times 25.(24^2 + 1)(25^2 + 1)(26^2 + 1)$

Again $5^2 \mid f(24)$

$n = 22,\ f(22) = \frac{1}{2} \times 21 \times 23 \times (22^2 + 1)(21^2 + 1)(23^2 + 1)$

$5 \mid 22^2 + 1,\ 5 \mid 23^2 + 1$

$\Rightarrow\ 5^2 \mid f(22)$

$n = 20,\ f(20) = \frac{1}{2}(19 \times 21)(20^2 + 1)(21^2 + 1)(19^2 + 1)$

Not divisible by any prime square

$\Rightarrow\ n = 20$

29. **(28)** $n^2 = 2^{38}.\ 3^{24}$

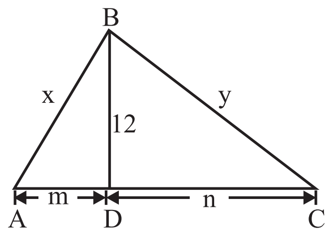
$\therefore$ Required number of divisors

$$= \frac{(38+1)(24+1)+1}{2} - 20 \times 13$$

$= 228$

$\therefore$ Last two digit = 28

30. **(25)**



$m^2 + 12^2 = x^2$

$n^2 + 12^2 = y^2$

$x^2 + y^2 = (m + n)^2$

$m^2 + 12^2 + n^2 + 12^2 = m^2 + n^2 + 2mn$

$2mn = 2.12^2$

$mn = 12^2$

for minimum value of $m + n$ such that $x, y \in I$

$m = 16,\ n = 9$ $(x = 20,\ y = 15)$

$\Rightarrow\ m + n = 25$