RMO (2024-2025)

(Non-KV & Non-JNV)

Time: 3 hours

INSTRUCTIONS

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, ..., a_n$ of 1, 2...., n nice if for every $k = 2, 3, ..., n$ we have that $a_1 + a_2 + ... + a_k$ is **not** divisible by k.

(a) If $n > 1$ is odd, prove that there is no nice rearrangement of 1, 2, ..., n.

(b) If n is even, find a nice rearrangement of 1, 2, ..., n.

2. For a positive integer n. Let R (n) be the sum of the remainders when n is divided by 1, 2, ..., n.

For example,

R(4) = 0 + 0 + 1 + 0 = 1,

R(7) = 0 + 1 + 1 + 3 + 2 + 1 + 0 = 8.

Find all positive integers n such that R(n) = $n - 1$.

3. Let ABC be an acute triangle with AB = AC. Let D be the point on BC such that AD is perpendicular to BC. Let O, H, G be the circumcentre, orthocentre and centroid of triangle ABC respectively. Suppose that 2.OD = 23. H.D. Prove that G lies on the incircle of triangle ABC.

4. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist i, j with $1 \le i < j \le 4$, such that $(a_i - a_j)^2 \le \frac{1}{5}$.

5. Let ABCD be a cyclic quadrilateral such that AB is parallel to CD. Let O be the circumcentre of ABCD, and L be the point on AD such that OL is perpendicular to AD. Prove that OB . (AB + CD) = OL . (AC + BD).

6. Let $n \ge 2$ be a positive integer. Call a sequence $a_1, a_2, ... a_k$ of integers an n-chain if $1 = a_1 < a_2 < ... < a_k = n$, and a_i divides a_{i+1} for all i, $1 \le i \le k - 1$. Let $f(n)$ be the number of n-chains where $n \ge 2$. For example, $f(4) = 2$ corresponding to the 4-chains {1, 4} and {1, 2, 4}. Prove that $f(2^m . 3) = 2^{m-1}(m + 2)$ for every positive integer m.

Hints & Solutions

1. For the first part, note that the given condition for $k = n$ implies that the sum $a_1 + a_2 + ... + a_n$ is not divisible by n. However, $a_1, a_2, ..., a_n$ is a rearrangement of 1, 2, ..., n so their sum is equal to $1 + 2 + ... + n = \frac{n(n+1)}{2}$ which is divisible by n for odd n. Thus, there cannot be any nice rearrangement of 1, 2, ..., n for odd n.

For the second part. let $n = 2m$. We show that the sequence

2, 1, 4, 3, 6, 5, 8, 7, ..., $2m$, $2m - 1$

is a nice rearrangement of 1, 2, ..., $2m$. For k even, we have $a_1 + a_2 + ... + a_k = \frac{k(k+1)}{2}$ which is not divisible by k since $\frac{(k+1)}{2}$ is not an integer. For k odd, we have

$a_1 + a_2 + ... + a_k = \frac{k(k+1)}{2} + 1$ which is 1 more than a multiple of k, so it is again not divisible by k for $k > 1$.

2. Let $n > 8$. The remainder when n is divided by some i satisfying $\frac{n}{2} < i \le n$ is $(n - i)$. Adding. we get that

$$n - 1 = R(n) \ge \sum_{i=\left[\frac{n}{2}\right]+1}^{n} (n-i) = \sum_{k=1}^{\left[\frac{n}{2}\right]-1} k$$

$$= \frac{1}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]-1\right) \ge \frac{1}{2}\left[\frac{n}{2}\right].4 \ge n$$

This is contradiction. So we get that n ≤ 8. Now we can compute that R(1) = R(2) = 0, R(3) = R(4) = 1, R(5) = 4, R(6) = 3, R(7) = R(8) = 8. Therefore the only solutions are $n = 1$ and $n = 5$.

3. Let I be the incentre of ΔABC. First note that O, G, H, I all lie on AD since it is simultaneously the perpendicular bisector of BC, the A-altitude, the A-median and the angle bisector of ∠BAC.

Suppose the reflection of H across BC is M. Then M lies on the circumcircle of ΔABC as well as lies on the angle bisector of ∠BAC, so it is the midpoint of arc BC not containing A. Then, we note that ∠MBI = ∠MIB, so MB = MI. Combining with MB = MC, we have that M is the circumcenter of ΔBIC.

Now, let the circumradius of ΔABC be R, let OD = x, HD = y, Then we have $x = \frac{23}{2}y$.

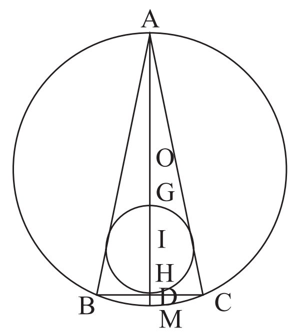
Also R = OM = OD + DM = OD + HD = $x + y$. Thus, $y = \frac{2}{25}$ R. This implies that AD = 2R – y = $\frac{48}{25}$R.

Now, recall that G divides AD in the ratio 2 : 1,

So, GD = $\frac{16}{25}$ R.

Also we have ΔMDB ~ ΔMBA since the angle at M is common and ∠MBD = ∠MAB, both, equalling $\frac{\angle BAC}{2}$.

Therefore, MB^2 = MD.MA. and hence



$MI^2 = MD \,.\, MA = y.\, 2R = \frac{4}{25}R^2 \Rightarrow MI = \frac{2}{5}R.$

Thus, ID = $\frac{8}{25}$ R, which combined with GD = $\frac{16}{25}$ R implies that GI = ID is equal to the inradius, proving that G lies on the incircle.

4. Let m be the minimum of $|a_i - a_j|$ over all $1 \le i < j \le 4$. Without loss of generality, we may assume that $a_1 \le a_2 \le a_3 \le a_4$. Then $a_j - a_i \ge (j - i)$ m for all $1 \le i \le j \le 4$. Thus,

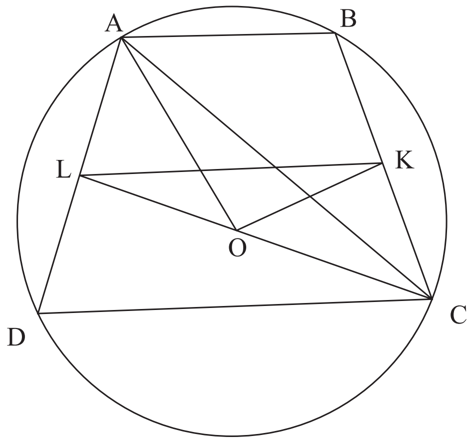
$$\sum_{1\le i<j\le 4} (a_i - a_j)^2 \ge \sum_{1\le i<j\le 4} (j-i)^2 m^2 = 20m^2.$$

On the other hand,

$$\sum_{1\le i<j\le 4} (a_i - a_j)^2 = 4(a_1^2 + a_2^2 + a_3^2 + a_4^2) - (a_1 + a_2 + a_3 + a_4)^2 \le 4.$$

Thus, $20m^2 \le 4 \Rightarrow m^2 \le \frac{1}{5}$.

5. Let K be the foot of perpendicular from O onto BC. Note that ABCD is a isosceles trapezium, therefore AC + BD = 2AC. We have that L and K are the midpoints of AD and BC respectively, therefore $LK = \frac{(AB+CD)}{2}$. Also OB = OA. Thus it suffices to prove that $\frac{OA}{AC} = \frac{OL}{LK}$.



Now ∠AOL = ∠ACD = ∠BDC = ∠COK. Thus ∠AOC = ∠LOK. Also note that OL = OK since distance from center to two equal chords is the same. Thus, ΔAOC and ΔLOK are isosceles triangles with ∠AOC = ∠LOK, hence they are similar, which immediately implies the desired.

6. We will prove for any two distinct primes p, q, that $f(p^m.p) = 2^{m-1}(m+2)$ for all integers $m \geq 1$. Suppose $n = p^m.q$, and let $\{a_1, a_2, ..., a_k\}$ be a n-chain. Then a_i divides a_{i+1} implies that $\frac{a_{i+1}}{a_i} = p^{b_i}.q^{c_i}$, where b_i, c_i are non-negative integers for $i = 1, ..., k-1$. Note that $a_{i+1} > a_i$ implies that b_i and c_i cannot be simultaneously 0.

Now, we have $b_1 + ... + b_{k-1} = m$ and $c_1 + ... + c_{k-1} = 1$. Thus, exactly one of the c_i will be equal to 1, and that implies that at most one of the b_i can be 0.

Recall that a composition of m is a sequence of positive integers adding to m. Corresponding to any l-length composition $x_1, ... x_1$ of m, we will get exactly $2l+1$ many n-chains. l of them are obtained by setting $b_i = x_i$ for all i and choosing one of $c_1 .... c_l$ to be 1, and rest to be 0. The other $l+1$ chains of length $l+1$ are obtained by choosing some $1 \leq j \leq l+1$, then setting $c_j = 1$, $b_j = 0$, $b_i = x_i$ for all $i < j$, and $b_i = x_{i-1}$ for all $i > j$.

This can be done in various ways as follows:

First way: it is well known that there are $\binom{m-1}{l-1}$ compositions of m with l parts. Therefore, we need

$$\sum_{l=1}^{m}\binom{m-1}{l-1}(2l+1) = \sum_{l=0}^{m}\binom{m-1}{l}(2l+3)$$

$$= 2\sum_{l=0}^{m-1} l.\binom{m-1}{l} + 3\sum_{l=0}^{m-1}\binom{m-1}{l}$$

$$= 2\left(\sum_{l=1}^{m-1}(m-1).\binom{m-2}{l-1}\right) + 3.2^{m-1}$$

$= 2(m-1)2^{m-2} + 3.2^{m-1} = 2^{m-1}(m+2)$.

Second way: We will show that the total number of compositions of m, is 2^{m-1} and the sum of the number of parts over all compositions of m is $2^{m-2}(m+1)$ via direct bijections. This finishes the problem since we get the sum of $(2l+1)$ over all compositions to be 2.

$2^{m-2}(m+1) + 2^{m-1} = 2^{m-1}(m+2)$.

For the first one, consider sequences of 0's and 1's such that there are exactly m 1's, no two 0's are adjacent and the sequence begins and ends with a 1. Then we can choose whether or not to insert a 0 in the m – 1 spaces between the 1's, hence there are 2^{m-1} possible ways to do it.

For the second one, we consider the above sequences but we put a single 0 at the end, and we also select a special 0. Then we can choose the special 0 first. If this is the last 0 then we get 2^{m-1} choices for the other zeroes, and if not then we have $m-1$ choices for the special 0, and then 2^{m-2} choices for the other spaces. This totals to $2^{m-2}(m+1)$ ways.