CHAPTER AT A GLANCE

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

1

Complex Numbers

If a and b are two real numbers, then a number of the form $z = a + ib$ such that $i^2 = -1$ or $i = \sqrt{-1}$ is called a complex number. Here a is the real part denoted by Re(z) & b is the imaginary part denoted by Im(z).
Suppose $z_1 = a + ib$ & $z_2 = c + id$, here $z_1 = z_2$ if $a = c$ & $b = d$.

2

Algebra of Complex Numbers

I. Addition : Let $z_1 = a + ib$, $z_2 = c + id$, then $z_1 + z_2 = (a+c) + i(b+d)$

II. Difference : $z_1 - z_2 = (a - c) + i(b - d)$

III. Multiplication : Let $z_1 = a + ib$, $z_2 = c + id$, then $z_1 z_2 = (ac - bd) + i(ad + bc)$

Properties of Multiplication:

(i) The product of any two complex numbers is a complex number.

(ii) $z_1 z_2 = z_2 z_1$ (Commutative Law)

(iii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (Associative law)

(iv) There exists the complex number $1 + i0$ (denoted as 1), called the multiplicative identity such that $z.1 = z$, for every complex number z.

(v) For every non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, we have the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z such that $z.\frac{1}{z} = 1$.

(vi) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributive law)

IV. Division : For two complex numbers z_1 & z_2, the quotient is given as :

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2} \quad (\text{where } z_2 \neq 0).$$

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Power of i

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, etc.
In general,
$i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
where, k is any integer.

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Identities

$(z_1 + z_2)^2 = z_1^2 + 2z_1 z_2 + z_2^2$

$(z_1 - z_2)^2 = z_1^2 - 2z_1 z_2 + z_2^2$

$(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$

$(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3$

$z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

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Modulus and Conjugate of a Complex Number

The modulus of $z = x + iy$, denoted by $|z|$ is the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$

The conjugate of $z = x + iy$, denoted by $\overline{z}$ where $\overline{z} = x - iy$

Properties :

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, |z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$ (iv) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}, \; z_2 \neq 0$

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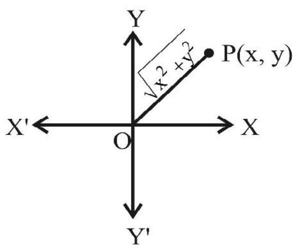
Argand Plane

The complex number $z = x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane. The plane having a complex number assigned to each of its point is called the argand plane or the complex plane.



x-axis is the real axis.
y-axis is the imaginary axis.

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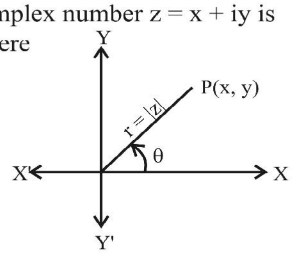
Polar Form of a Complex Number

The polar form of the complex number $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$, where

$r = \sqrt{x^2 + y^2} = |z|$

$\cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$



θ is termed as argument or amplitude of z denoted by arg z.

The value of θ, such that $-\pi < \theta \leq \pi$ is called the principal argument of z.

Euler's Form of a Complex Number

$\because e^{i\theta} = \cos\theta + i\sin\theta \; \therefore z = re^{i\theta}$

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Quadratic Equations

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real numbers, $a \neq 0$, $b^2 - 4ac < 0$ are given by

$$x = \frac{-b \pm \sqrt{4ac - b^2}\, i}{2a}$$

A polynomial equation of n degree has n roots.

Relation between roots and coefficient

sum roots $(\alpha + \beta) = -\frac{b}{a}$

Product of roots $(\alpha \cdot \beta) = \frac{c}{a}$

CHAPTER AT A GLANCE

CONIC SECTIONS

1. Circle

A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

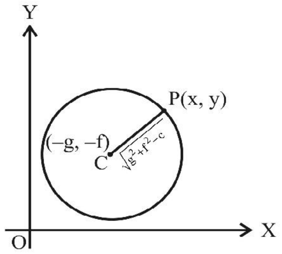
The fixed point is called the centre of the circle and the distance from the centre to a point on the circle is called the radius of the circle.

2. General Equation of a Circle

The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where g, f, c are constants.

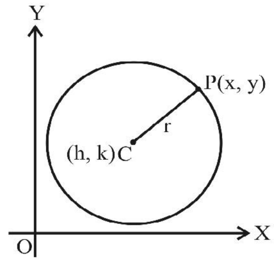
Centre of the circle is $(-g, -f)$

Radius of the circle is $\sqrt{g^2 + f^2 - c}$



3. Standard Equation of a Circle

The equation of a circle having centre (h, k) & radius r is $(x-h)^2 + (y-k)^2 = r^2$



Note that if (x_1, y_1) and (x_2, y_2) be the extremities of a diameter, then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

4. Position of a Point w.r.t Circle

(i) If the distance of a point from the centre of the given circle is greater than the radius of the circle, then the point lies outside the circle.

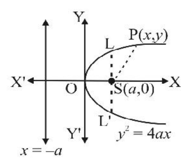
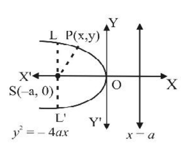
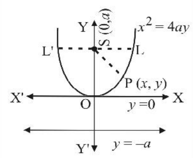
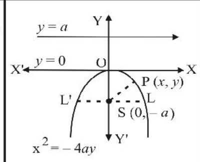
(ii) If this distance is less than the radius of the circle, then the point lies inside the circle.

5. Parabola

A parabola is a set of all points in a plane that are equidistant from a fixed line & a fixed point in a plane.

The fixed line is called the directrix of the parabola and the fixed point is called the focus.

6. Four Standard Forms of the Parabola

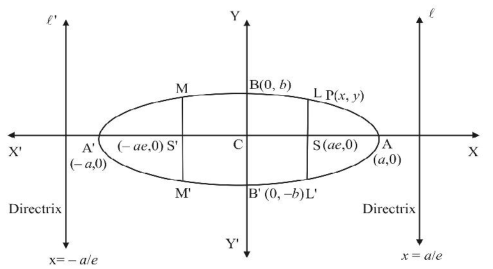
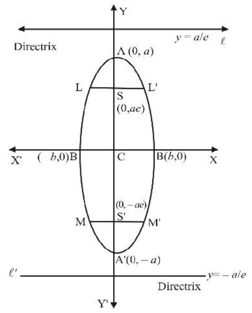
Standard Equation	$y^2 = 4ax\ (a > 0)$	$y^2 = -4ax\ (a > 0)$	$x^2 = 4ay\ (a > 0)$	$x^2 = -4ay\ (a > 0)$
Shape of Parabola				
Vertex	O (0, 0)	O (0, 0)	O (0, 0)	O (0, 0)
Focus	S $(a, 0)$	S $(-a, 0)$	S $(0, a)$	S $(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$

7. Ellipse

An ellipse is the set of all points in a plane, the sum of whose distances, form two fixed points in the plane is a constant.

The two fixed points are called the 'foci' of the ellipse.

Two Standard Forms Of The Ellipse

Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b), where a and b are constants (Horizontal Form of an Ellipse)	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ (a > b), where a and b are constants (Vertical Form of an Ellipse)
Shape of the ellipse		
Centre (c)	(0, 0)	(0, 0)
Equation of major axis(AA')	y = 0	x = 0
Equation of minor axis(BB')	x = 0	y = 0
Length of major axis(=AA')	$2a$	$2a$
Length of minor axis(=BB')	2b	2b
Foci (S and S')	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices (A and A')	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices (ℓ and ℓ')	$x = \pm a/e$	$x = \pm a/e$
Eccentricity(e)	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum (LL' or MM')	$2b^2/a$	$2b^2/a$

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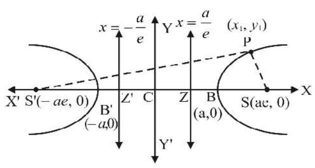
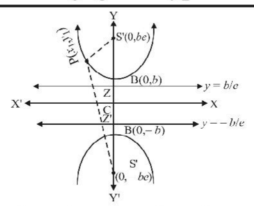
Hyperbola

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.

The two fixed points are called foci of the hyperbola.

Hyperbola and its Conjugate

	Hyperbola	Conjugate Hyperbola
		
Standard Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Eq. of transverse axis	y=0	x=0
Eq. of conjugate axis	x–0	y–0
Length of transverse axis	$2a$	2b
Length of conjugate axis	2b	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$y = \pm a/e$	$y = \pm b/e$
Vertices	$(+a, 0)$	$(0, +b)$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{b^2}}$
Length of latus rectum	$2b^2/a$	$2a^2/b$

CHAPTER AT A GLANCE

MATRICES

1. Matrix

A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements of the matrix.

For example $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ is a matrix.

The horizontal lines of elements in the above matrix are said to constitute, **rows** of the matrix & vertical lines of elements are said to constitute **columns** of the matrix. Thus above matrix has 2 rows and 3 columns.

2. Order of a Matrix

A matrix having m rows and n columns is called a matrix of order m×n or simply m×n matrix.
or $A = [a_{ij}]_{m\times n}, 1 \le i \le m, 1 \le j \le n$ $i, j \in N$
a_{ij} is an element lying in the i^{th} row & j^{th} column.
The number of elements in m×n matrix will be mn.

3. Types of Matrix

(i) **Column Matrix :** A matrix is said to be a column matrix if it has only one column, i.e., $A = [a_{ij}]_{m\times 1}$ is a column matrix of order m×1.

(ii) **Row Matrix :** Row matrix has only one row, i.e., $B = [b_{ij}]_{1\times n}$ is a row matrix of order 1×n.

(iii) **Square Matrix :** Square matrix has equal number of rows and columns, i.e., $A = [a_{ij}]_{m\times m}$ is a square matrix of order m.

(iv) **Diagonal Matrix :** A square matrix is said to be diagonal matrix if all of its non-diagonal elements are zero, i.e., $B = [b_{ij}]_{m\times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, where $i \ne j$.

(v) **Scalar Matrix :** It is a diagonal matrix with all its diagonal elements equal, i.e., $B = [b_{ij}]_{m\times n}$ is a scalar matrix if
$b_{ij} = 0$, where $i \ne j$
$b_{ij} = k$, when $i = j$ & k = constant.

(vi) **Identity Matrix :** It is a diagonal matrix having all its diagonal elements equal to 1, i.e., $A = [a_{ij}]_{m\times n}$ is an identity matrix if

$$a_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \ne j \end{cases}$$

we denoted identity matrix by I_n when order is n.

(vii) **Zero Matrix :** A matrix is said to be zero or null matrix if all its elements are zero. It is denoted by O.

4. Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if
(i) they are of the same order
(ii) each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for all i & j

5. Addition of Matrices

Sum of the two matrices is a matrix obtained by adding the corresponding elements of the given matrices, i.e., $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order m×n. Then sum of two matrics A & B is defined as $C = [c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$ for all i & j.

Difference of matrices : The difference A – B is defined as $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$ for all i & j. In order words D = A – B = A + (–B), that is the sum of matrices A & (–B).

Properties of matrix Addition

(i) Commulative Law: A + B = B + A

(ii) Associative Law: (A + B) + C = A + (B + C)

(iii) Existance of Additive Identity: Let $A = [a_{ij}]_{m\times n}$ & O = zero matrix of order m×n, then A + O = O + A = A. Here O is the additive identity for matrix addition.

(iv) Existance of Additive Inverse
Let $A = [a_{ij}]_{m\times n}$ be any matrix then we have another matrix as $-A = [-a_{ij}]_{m\times n}$ such that A + (–A) = (–A) + A = O. Here –A is the additive inverse of A or negative of A.

6. Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m\times n}$ be a matrix & k be a number.
Then, $kA = Ak = [ka_{ij}]_{m\times n}$

Properties
(I) k(A + B) = kA + kB (ii) (k + t) A = kA + tA.

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Multiplication of Matrices

If A & B are any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B. If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$, then their product $AB = C = [c_{ij}]$, is a matrix of order m×p, where (ij)th element of $AB = C_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}$

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Properties of Matrix Multiplication

(i) Associative Law for Multiplication : If A, B & C are three matrices of order m×n, n×p & p×q respectively, then (AB) C = A(BC)

(ii) Distributive Law : For three matrices A, B & C
(a) A(B + C) = AB + AC
(b) (A + B) C = AC + BC, whenever both sides of equality are defined.

(iii) Matrix Multiplication is not commutative in general, i.e., $AB \neq BA$ (in general).

(iv) Existence of Multiplicative Identity : For every square matrix A, there exists an identity matrix I of same order such that IA = AI = A.

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Transpose of a Matrix

The matrix obtained from a given matrix A by changing its rows into its corresponding columns or columns into its corresponding rows is called transpose of matrix A & it is denoted by A^T or A'. If the order of A is m × n, then order of A^T is n×m.
In other words if $A=[a_{ij}]_{m\times n}$ then $A^T=[a_{ji}]_{n\times m}$

Properties of Transpose of the Matrices

For any matrices A & B of suitable orders, we have:
(i) $(A^T)^T = A$
(ii) $(kA)^T = k(A)^T$ (where k is constant)
(iii) $(A+B)^T = A^T + B^T$
(iv) $(AB)^T = B^T A^T$

Symmetric & Skew Symmetric Matrices

Symmetric Matrix
A square matrix $A=[a_{ij}]$ is called a symmetric matrix, if $a_{ij}=a_{ji}$ for all i, j or $A^T = A$

Skew Symmetric Matrix
A square matrix $A=[a_{ij}]$ is called a skew-symmetric matrix, if $a_{ij}=-a_{ji}$ for all i, j or $A^T = -A$.

Properties of Symmetric & Skew Symmetric Matrices
(I) For any square matrix A with real number entries, $(A+A^T)$ is a symmetric matrix & $(A-A^T)$ is a skew symmetric matrix.
(ii) Any square matrix A can be expressed as the sum of a symmetric & a skew symmetric matrix as

$$A=\left[\frac{1}{2}(A+A^T)\right]+\left[\frac{1}{2}(A-A^T)\right]$$

Invertible Matrix and Inverse Matrix

If A is a square matrix and there exists another square matrix B of the same order such that AB = BA = I, then B is called the inverse matrix of A & it is denoted by A^{-1}.
In that case A is said to be invertible matrix.

Properties of Invertible Matrices
(i) Uniqueness of Inverse : Inverse of a square matrix, if it exists, is unique.
(ii) $(AB)^{-1} = B^{-1}A^{-1}$

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Inverse of a Matrix by Elementary Operations

If A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write A = IA & apply a sequence of row operations on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, we write A = AI & apply a sequence of column operations on A = AI till we get, I = AB.

Elementary Operation (Transformation of a Matrix)

There are six operations on a matrix, three of which are due to rows & three due to columns, called elementary operations or Transformations.
(i) The interchange of any two rows or two columns symbolically, interchange of i^{th} & j^{th} rows is denoted by $R_i \leftrightarrow R_j$ & same will be for columns, i.e., $C_i \leftrightarrow C_j$.
(ii) The multiplication of the elements of any row or column by a non zero number. For rows it is denoted as $R_i \leftrightarrow kR_i$, $k \neq 0$ & for columns: $C_i \leftrightarrow kC_i$.
(iii) The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number. Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted as: $R_i \leftrightarrow R_i + kR_j (k \neq 0)$
For columns : $C_i \leftrightarrow C_i + kC_j$

CHAPTER AT A GLANCE

APPLICATION OF DERIVATIVES

1. Rate of Change of Quantities

The rate of change of y with respect to x at a point $x = x_0$ is given by $\left(\frac{dy}{dx}\right)_{x=x_0}$

Note that $\frac{dy}{dx}$ is positive if y increases with increase in x and is negative if y decreases with increase in x.

2. Increasing and Decreasing Functions

(1) (I) Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in I$.

(ii) strictly increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in I$.

(iii) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \ge f(x_2)\ \forall\ x_1, x_2 \in I$.

(iv) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in I$.

(II) A function f is said to be increasing at x_0 if there exists an interval $I = (x_0 - h, x_0 + h), h > 0$ such that for x_1, x_2

$x_1 < x_2$ in $I \Rightarrow f(x_1) \le f(x_2)$

Similarly, the other cases i.e., strictly increasing, decreasing and strictly decreasing can be clarified.

(2) A function f(x) defined in the interval [a,b] will be

Monotonic increasing $\Leftrightarrow f'(x) \ge 0\ x \in (a, b)$

Monotonic decreasing $\Leftrightarrow f'(x) \le 0\ x \in (a, b)$

Constant function $\Leftrightarrow f'(x) = 0\ x \in (a, b)$

Strictly increasing $\Leftrightarrow f'(x) > 0\ x \in (a, b)$

Strictly decreasing $\Leftrightarrow f'(x) < 0\ x \in (a, b)$

Properties of Monotonic Functions

(1) If f (x) and g(x) are monotonically (strictly) increasing (decreasing) functions on [a, b], then gof (x) is a monotonically (strictly) increasing function on [a, b].

(2) If one of the two functions f (x) and g(x) is strictly (monotonically) increasing and other is strictly (monotonically) decreasing, then gof (x) is strictly (monotonically) decreasing on [a, b].

3. Tangents and Normals

- The equation of the tangent at (x_0, y_0) is given below:

  $y - y_0 = m(x - x_0)$,

  where m = slope of tangent $= \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$ or $f'(x_0)$

- The equation of the normal at (x_0, y_0) is given below:

  $y - y_0 = -\frac{1}{m}(x - x_0)$,

  where m = slope of tangent at (x_0, y_0)

4. Approximations

Let y = f(x), Δx be a small increment in x & Δy be the increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x$$

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Test of Local Maxima & Minima

First Derivative Test:
Let f(x) be a function differentiable at x = a. Then

(a) x = a is a point of local maximum of f(x), if
- (i) $f'(a) = 0$ and
- (ii) $f'(x)$ changes sign from positive to negative as x increases through a

(b) x = a is a point of local minimum of f(x), if
- (i) $f'(a) = 0$ and
- (ii) $f'(x)$ changes sign from negative to positive as x increases through a

(c) If $f'(a) = 0$, but $f'(x)$ does not change sign as x increases through a, that is $f'(a)$ has the same sign in the complete neighourhood of a, then a is neither a point of local maximum nor a point of local minimum. In this case, x = a is a point of inflection.

Second Derivative Test:
Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then
- (i) x = c is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$
  The value f(c) is local maximum value of f.
- (ii) x = c is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$
  In this case, f(c) is local minimum value of f.
- (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$
  In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflection.

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Maxima and Minima

1. Let f be a funciton defined on an interval I. Then
   - (a) f is said to have a maximum value in I. if there exists point c in I such that $f(c) \geq f(x)$, for all $x \in I$.
     f(c) is the maximum value and point c is a point of maximum value of f in I.
   - (b) f is said to have a minimum value in I. if there exists a point c in I such that $f(c) \leq f(x)$, for all $x \in I$.
     f(c) is the minimum value and point c is a point of minimum value of f in I.
   - (c) f is said to have an extreme value in I if there exists a point c in I such that f(c) is either a maximum value or a minimum value of f in I.
     f(c) is an extreme value and point c is called an extreme point.
2. Let f be a real valued function and let c be an interior point in the domain of f. Then
   - (a) c is called a point of local maxima if there is an $h > 0$ such that
     $f(c) \geq f(x)$, for all x in $(c - h, c + h)$
     The value f(c) is called the local maximum value of f.
   - (b) c is called a point of local minima if there is an $h > 0$ such that
     $f(c) \leq f(x)$, for all x in $(c - h, c + h)$
     The value f(c) is called the local minimum value of f.
3. Let f be a function defined on an open interval I. Suppose $c \in I$ be any point. If f has a local maxima or a local minima at x = c, then either $f'(c) = 0$ or f is not differentiable at c.

Absolute Maxima & Absolute Minima

Let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.
Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then
- (i) $f'(c) = 0$ if f attains its absolute maximum value at c.
- (ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

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Steps for Finding Absolute Maxima and/or Absolute Minima

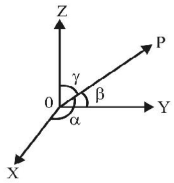
- (i) Find all critical points of f in the interval, i.e., find value of x where either $f'(x) = 0$ or f is not differentiable.
- (ii) Take the end points of the interval.
- (iii) At all the above points (in step (i) and (ii)) calculate the value of f.
- (iv) Identify the maximum and minimum values of f out of the values calculated in step (iii). The maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

CHAPTER AT A GLANCE

THREE DIMENSIONAL GEOMETRY

1 Direction Cosines of a Line (DC's)

The direction cosines are generally denoted by l, m, n.



Hence, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Note that $l^2 + m^2 + n^2 = 1$

2 Direction Ratios of a Line (DR's)

Any three numbers a, b and c proportional to the direction cosines l, m and n, respectively are called direction ratios of the line.

- The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$
- $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$
- $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

3 Equation of a Line

1. Equation of a line through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda\vec{b}$

In cartesian form,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Here, a, b, c are also the direction ratios of the line.

2. Equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

In cartesian form,

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

4 Angle Between Two Lines

In vector form,

The angle between two lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

In cartesian form,

The angle between two lines :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is :

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

- If two lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$
- If two lines are parallel, then $\vec{b}_1 = \lambda\vec{b}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

5 Shortest Distance Between Two Lines

1. Distance Between Parallel Lines

The shortest distance between parallel lines $L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}$ is

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

2. Distance Between Two Skew Lines

In vector form,

The distance between two skew lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

In cartesian form,

The distance between two skew lines :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is :

$$d = \left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

6 Equation of a Plane in Normal Form

Vector Form

$\vec{r} \cdot \hat{n} = d$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{n}$ is the unit vector along the normal from origin to the plane.

d is perpendicular distance of the plane from the origin.

Cartesian Form

$lx + my + nz = d$

where l, m, n are the direction cosines of $\hat{n}$ (unit vector along the normal from origin to the plane).

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Equation of a Plane Perpendicular to a Given Vector and Passing Through a Given Point

Vector Form
Let a plane pass through a point with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$. Then its equation is given as: $(\vec{r}-\vec{a})\cdot\vec{N}=0$

Cartesian Form
Let a plane pass through a point (x_1, y_1, z_1) & the direction ratio of the vector perpendicular to the plane be A, B, C. Then its equation is given as:

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

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Equation of a Plane Passing Through Three Non-Collinear Points

Vector Form

$$[\vec{r}\,\vec{b}\,\vec{c}]+[\vec{r}\,\vec{a}\,\vec{b}]+[\vec{r}\,\vec{c}\,\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

or $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$

where, $\vec{a},\vec{b},\vec{c}$ are the position vector of three given non-collinear points through which the plane passes.

Cartesian Form
The equation of plane passing through three non-collinear points Y with coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) & (x_3, y_3, z_3) is given as:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

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Intercept Form of the Equation of a Plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

where a, b, c are the intercepts made by the plane on x, y & z axes respectively.

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Plane Passing Through the Intersection of Two Given Planes

Vector Form
Equation of plane passing through the point of intersection of two planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is given as:

$$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$$

Cartesian Form
Let
$\vec{n}_1=A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$
$\vec{n}_2=A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$
and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$,
therefore its cartesian equation is:

$$(A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0$$

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Coplanarity of Two Lines

Vector Form
Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$

are coplanar, if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

Cartesian Form
Two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

are coplanar, if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

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Angle Between a Line and a Plane

Vector Form
Angle between a line $\vec{r}=\vec{a}+\lambda\vec{b}$ and a plane $\vec{r}\cdot\vec{n}=d$ is

$$\cos\theta=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Cartesian Form
Angle between a line $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and a plane $a_2x+b_2y+c_2z=d$ is given as:

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If line is perpendicular to the plane, then $\vec{n}=\lambda\vec{b}$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
- If line is parallel to the plane, then $\vec{n}\cdot\vec{b}=0$ or $a_1a_2+b_1b_2+c_1c_2=0$

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Distance of a Point from a Plane

Vector Form
Distance of a point with position vector $\vec{a}$ from a plane $\vec{r}\cdot\vec{n}=d$ is given as:

$$\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$$

Cartesian Form
Distance of a point (x_1, y_1, z_1) from a plane : $ax+by+cz=d$ is given as :

$$\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

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Angle Between Two Planes

Vector Form : The angle between two planes $\vec{r}\cdot\vec{n}=d_1$ & $\vec{r}\cdot\vec{n}=d_2$ is given as:

$$\cos\theta=\left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right|$$

Cartesian Form The angle between two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given as

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If two planes are perpendicular, then $\vec{n}_1\cdot\vec{n}_2=0$ or $a_1a_2+b_1b_2+c_1c_2=0$
- If two planes are parallel, then $\vec{n}_1=\lambda\vec{n}_2$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$